

ÉCOLE POLYTECHNIQUE

NON CONFIDENTIAL REPORT

Uncertainty and the Le Châtelier Principle : A Lattice Theory Perspective

---

Aditya Srivastava
Master in Economics (M1)

Institut Polytechnique de Paris 2024-25

Internship Tutor : Prof. Timothy Van Zandt, INSEAD

Referent Professor : Prof. Yves Le Yaouanq, CREST - IP

Internship Dates : April 1, 2025 - September 15, 2025

Declaration of Academic Integrity

Hereby I, Aditya Srivastava, confirm that:

1. The results presented in this report are my own work.
2. I am the author of this report
3. I have not used the work of others without clearly acknowledging it, and quotations and paraphrases from any source are clearly indicated.

Aditya Srivastava, July 10, 2025

Signature



Abstract

The Le Châtelier principle captures the fundamental economic insight that short-run responses to shocks are typically attenuated relative to long-run adjustments due to binding constraints. Originally formalized by Paul Samuelson under differentiability and local assumptions, this principle was later extended globally using lattice-theoretic methods that accommodate discrete, non-smooth, and strategic settings. Despite these advances, the interaction between dynamic adjustment processes and risk preferences under uncertainty remains underexplored. This paper investigates how stochastic shocks and risk aversion affect monotone comparative statics and the attenuation of responses over time, extending recent dynamic frameworks with adjustment costs. The results aim to deepen understanding of economic behavior in realistic settings where both friction and uncertainty are key.

Contents

1 Introduction **4**
1.1 Background . . . 4
1.2 Related Literature . . . 5

2 Le Châtelier-Samuelson Principle: A Classical Exposition **7**
2.1 Formal Setup and General Result . . . 7
2.1.1 Semi-Formal Proof . . . 8
2.1.2 Application: Labor Demand in the Short Run vs. Long Run . . . 9
2.2 Limits of the Le Châtelier-Samuelson Principle . . . 10
2.2.1 Example: Non-monotonic labor demand under fixed capital . . . 11

3 A Global Le Châtelier Principle **11**
3.1 Feedback Effects and the Limits of Local Reasoning . . . 11
3.2 Supermodularity and the Architecture of Positive Feedback . . . 13
3.3 Extending Local Samuelson Insights . . . 15
3.4 Global Le Châtelier Principle: Supermodular Case . . . 16
3.5 Weak Form Global Principle . . . 17
3.6 Jamison's Lattice-Theoretic Generalization . . . 17

4 Comparative Statics Under Uncertainty **18**
4.1 Comparative Statics of a Risk-neutral Firm facing Output Price Uncertainty . . . 19
4.1.1 Extensions . . . 21
4.2 Risk-Averse Firm Under Input Price Uncertainty . . . 21
4.2.1 Constant Absolute Risk Aversion (CARA) . . . 22
4.2.2 Decreasing Absolute Risk Aversion (DARA) . . . 22
4.2.3 Increasing Absolute Risk Aversion (IARA) . . . 23
4.3 Monotone Likelihood Ratio (MLR) Improvements in Price Uncertainty . . . 23
4.3.1 Why Focus on MLR Improvements? . . . 24
4.4 Increases in Risk Aversion . . . 25
4.5 Discussion . . . 25

5 A Strong Le Châtelier Principle under Uncertainty and Adjustment Costs **26**
5.1 Comparative Benchmark: Long-Run Frictionless Optimum . . . 26
5.2 Dynamic Problem and Main Results . . . 27
5.3 Significance of Dynamic Results . . . 34
5.4 Connection to Dekel et al. (2025) . . . 35

6 On the Global Convexity of Value of Information **36**
6.1 Setup and Notation . . . 37
6.2 Reformulation as a Constrained Problem . . . 38
6.3 Concluding Remarks on Global Convexity . . . 42

7 Conclusion and Potential Extensions **43**

1 Introduction

1.1 Background

The Le Châtelier principle originated in chemistry, where it was first introduced by French scientist Henri Louis Le Châtelier in the late 19th century to describe how systems at equilibrium respond to external shocks. His core idea was that if an external condition, such as temperature, pressure, or concentration, changes, a system will adjust internally in a way that tends to counteract the effect of that disturbance. Although the principle became firmly established in classical thermodynamics, its verbal formulation remained somewhat vague and heuristic in operational terms. Later summaries, especially in physics and economics, emphasized its core behavior: systems tend to resist change and move in the direction that offsets external pressure.

In economics, Paul Samuelson gave this principle a rigorous mathematical foundation, showing that it naturally follows from optimization behavior under constraints. Building on a suggestion from his mentor at Harvard University, E.B. Wilson, Samuelson demonstrated that when a maximizing agent faces a shock, say, in prices or income, their short-run adjustment, constrained by fixed factors, will typically be smaller than their long-run response once those constraints are relaxed. This insight, developed in his Foundations of Economic Analysis (1947) and extended in later works, linked the Le Châtelier Principle to core tools of comparative statics and the envelope theorem. Importantly, Samuelson clarified that there are really two related ideas at play: one concerns the direction of first-order responses to shocks; the other—now most closely associated with the Le Châtelier principle—concerns how the imposition or relaxation of constraints alters the magnitude of those responses.

However, this early economic formulation was mostly local, relying on differentiability and second-order conditions. It applied well to smooth systems but was limited when dealing with discrete decisions, inequality constraints, or strategic environments. Several attempts in the 1970s and 1980s extended the principle to more general settings—using bordered Hessians, matrix perturbations, and nonlinear systems—but these still depended on differentiability or convexity. The goal of a truly global version, applicable in non-smooth or discrete environments, remained largely out of reach until the 1990's.

This changed with the development of lattice-theoretic approaches to comparative statics, most notably in the work of Milgrom and Roberts (1996). By invoking ordinal complementarity and supermodularity, they established a global version of the Le Châtelier Principle that holds even without differentiability, convexity, or smoothness. Later, Dekel et al. (2025) extended these ideas to dynamic environments with adjustment costs, showing that long-run responses continue to dominate short-run ones under natural monotonicity assumptions.

Despite these advances, a crucial dimension remains underexplored: how uncertainty and risk preferences interact with dynamic adjustment processes. While monotone comparative statics under uncertainty has been studied—particularly by Athey (2002) and Suen et al. (2000)—most models consider static or one-shot environments. Even where adjustment costs are modeled dynamically, uncertainty typically enters only through asymmetric cost functions rather than preferences. This report aims to bridge that gap. Building on the dynamic framework of Dekel et al. (2025), I investigate how the presence of risk preferences and stochastic shocks affects monotonicity and the structure of short-run versus long-run responses. The broader objective is to identify conditions under which a dynamic Le Châtelier Principle continues to hold in environments where both frictions and uncertainty are central features of decision-making.

1.2 Related Literature

The economic interpretation of the Le Châtelier principle originated with Paul Samuelson, who formally introduced it in his book Foundations of Economic Analysis (1947) and extended it in later works such as "An Extension of the Le Châtelier Principle" (1960) and his RAND monograph on linear programming (1949). Specifically, in his 1960 paper, Samuelson used Jacobi's well-known theorem of reciprocal determinants (not as well known as it is usually needed to find a reference; See Aitken (1933)) to corroborate that short-run responses to shocks are attenuated compared to long-run adjustments under usual sufficient second-order conditions. Broadly, his contributions extend beyond these foundational pieces: His collected papers include work on comparative statics (e.g., "Comparative Statics and Dynamics"; "The Relation between Hicksian Stability and True Dynamic Stability"), and his review of the heuristic connection between chemical and economic equilibria in later reflections (Samuelson, 1972, 1963).

Subsequent economists broadened and deepened this insight. Kusumoto (1976, 1977) and Fujimoto (1980) offered global, differentiable versions of the principle as they contributed significantly to the literature on Leontief models. Eichhorn and Oettli (1972) provided an abstract general formulation. These works explored comparative statics in linear and non-linear settings, using tools like bordered Hessians and matrix perturbations. Dietzenbacher (1992) revisited Samuelson's core message in the context of input-output models. Lady and Quirk (2004) contributed by analyzing non-maximizing systems through a theory of cycles in determinants, proving that the Le Châtelier principle applies to a broad class of systems identified earlier by Morishima (1952), which incorporate both substitutes and complements. This result extends the applicability of the principle beyond classical maximizing frameworks, highlighting its robustness in more general economic environments.

Eugene Silberberg further contributed substantially to applications of the generalized envelope theorem and comparative statics in Silberberg (2018). He generalized Samuelson's local result to hold for own-price effects and globally well-behaved functions. Hoel and Vislie (1983) identified important examples where the principle may be violated, es-

pecially when uncertainty does not stem from adjustment costs. These case studies shed light on the subtleties involved in extending the classical results.

The shift from local to global comparative statics was crystallized by Milgrom and Roberts (1996), who recast the Le Châtelier Principle via supermodularity hypothesis and lattice-theoretic methods. This qualitative approach was part of a broader development in comparative static theory of optimization on lattices, first initiated by Topkis (1978) and continued by Milgrom and Shannon (1994), Shannon (1995). This framework substantially generalized the classes of models for which comparative statics results could be developed. Athey (2002); Athey and Schmutzler (1995) established conditions for monotonic responses under uncertainty. Building on this foundation, Jamison (2006) extended the results of Milgrom, developing a lattice theoretic framework that generalized the principle to hold for the ordinal complementarity assumptions about the objective function. Quah and Strulovici (2009) further applied the framework to constrained optimization and dynamic models with incomplete information.

In addition to optimization problems, the Le Châtelier principle has been studied in strategic settings. Alexandrov and Bedre-Defolie (2017) analyzed how shocks propagate in games and market systems, characterizing pass-through and attenuation effects in equilibrium. His results suggest that the principle can be meaningfully applied to multi-agent settings with strategic interdependence, further broadening its scope.

Finally, Dekel et al. (2025) developed one of the most general dynamic formulations of the Le Châtelier principle: they show that in models with adjustment costs and ordinal complementarity, long-run responses dominate short-run ones, and optimal trajectories are monotonic . However, their framework treats uncertainty mainly as cost asymmetry, not as a feature of preference. Meanwhile, comparative statics under risk aversion, such as Pindyck's investment under uncertainty, Athey's uncertainty-robust monotonicity, and Suen et al. (2000)'s static treatment of risk, illustrate how risk preferences can change response patterns, although typically in models of one period.

In a related but earlier contribution, Plummer (1985) established that, in a neighborhood of a point where uncertainty acts as a binding constraint, there exists a local positive relationship between uncertainty and the value of information. He further speculated that conditions sufficient for a global Le Châtelier principle might also ensure a globally positive relation between uncertainty and information value. This observation raises an open question: could lattice-theoretic or monotone comparative statics techniques provide a foundation for such a global connection? While this link remains largely unexplored, it suggests that tools developed for global comparative statics might help formalize deeper relationships between uncertainty, information, and economic responses.

Collectively, this literature underscores a persistent gap: while dynamic adjustment models capture frictions and long-run effects, they seldom integrate risk-averse preferences dynamically. My report seeks to bridge this gap, asking whether the Le Châtelier principle and monotone comparative statics can be preserved when both adjustment costs and uncertainty shape economic decisions.

2 Le Châtelier-Samuelson Principle: A Classical Exposition

Consider a firm minimizing cost or a consumer maximizing utility. In the long run, the decision-maker can adjust all variables (e.g., capital and labor); in the short run however, some variables are fixed. Suppose a price or income shock occurs. Because short-run choices are restricted by the fixed variables, the agent cannot adjust as much as in the long run. Thus, short-run responses are typically smaller in magnitude — a pattern echoed across microeconomic behavior, from labor demand to intertemporal consumption to monopoly pricing under capacity constraints.

Samuelson's contribution was to rigorously demonstrate this comparative statics result using the tools of classical optimization theory, applying the theory of negative definite matrices, bordered Hessians, and Jacobi's Theorem of determinants. This result is now referred to as the Le Châtelier-Samuelson principle.

2.1 Formal Setup and General Result

Let the objective function be:

$$F(x_1, \ldots, x_n, \theta) = f(x_1, \ldots, x_n) - \sum_{i=1}^{n} \theta_i x_i,$$

where f is strictly concave and twice continuously differentiable. The vector $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)$ represents parameters (e.g., prices or marginal costs). The first-order conditions (FOCs) for a maximum are:

$$\frac{\partial f}{\partial x_i}(x^*) = \theta_i \quad \text{for } i = 1, \ldots, n.$$

Let $[H] = [f_{i,j}]$ denote the Hessian matrix of second derivatives:

$$[H] = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right],$$

and assume $[H]$ is negative definite, so x^* is a strict maximum.

We now analyze how the optimal solution $x^*(\theta)$ changes in response to a marginal change in θ_i, and how this response is affected when additional linear equality constraints are imposed on the problem. The result is formalized in the following theorem:

Theorem 1 (Le Châtelier-Samuelson Principle: General Form)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be strictly concave and twice continuously differentiable, and let $x^*(\theta)$ solve the unconstrained problem defined above. Suppose we introduce r linear equality constraints of the form $\sum_{j=1}^{n} g_j^{\beta}(x_j - x_j^0) = 0$ for $\beta = 1, \ldots, r$, satisfied at a baseline point x^0. Let $x_r^*(\theta)$ be the solution to the constrained problem. Then under standard second-order conditions,*

$$\left(\frac{\partial x_i}{\partial \theta_i}\right)_0 \leq \left(\frac{\partial x_i}{\partial \theta_i}\right)_1 \leq \cdots \leq \left(\frac{\partial x_i}{\partial \theta_i}\right)_r < 0,$$

where the subscript r denotes the number of constraints imposed.

2.1.1 Semi-Formal Proof

We now present a detailed exposition of this result, following Samuelson's classical approach using Cramer's Rule, and Jacobi's theorem on determinants.

Unconstrained Case ($r = 0$). Differentiating the first-order conditions with respect to θ_j:

$$\sum_{k=1}^{n} f_{ik}(x^*) \frac{\partial x_k}{\partial \theta_j} = \delta_{ij},$$

where δ_{ij} is the Kronecker delta. This linear system can be solved using Cramer's Rule. In particular:

$$\frac{\partial x_i}{\partial \theta_i} = \frac{H_{ii}}{|H|}, \tag{1}$$

where H_{ii} is the $(n-1) \times (n-1)$ principal minor of H obtained by deleting the i-th row and column. Since H is negative definite, this implies $\frac{\partial x_i}{\partial \theta_i} < 0$.

Constrained Case ($r \geq 1$). Now suppose we impose r linear constraints:

$$\sum_{j=1}^{n} g_j^{\beta}(x_j - x_j^0) = 0, \quad \text{for } \beta = 1, \ldots, r,$$

where x^0 is the equilibrium point that satisfies these constraints.

We introduce Lagrange multipliers λ_β and define the augmented Lagrangian:

$$\mathcal{L}(x, \lambda) = f(x) - \sum_{i=1}^{n} \theta_i x_i + \sum_{\beta=1}^{r} \lambda_\beta \left(\sum_{j=1}^{n} g_j^{\beta}(x_j - x_j^0) \right).$$

The first-order conditions now become:

$$\frac{\partial f}{\partial x_i}(x) + \sum_{\beta=1}^{r} \lambda_\beta g_i^{\beta} = \theta_i, \quad (i = 1, \ldots, n), \tag{2}$$

$$\sum_{j=1}^{n} g_j^{\beta}(x_j - x_j^0) = 0, \quad (\beta = 1, \ldots, r). \tag{3}$$

This defines a system of $n+r$ equations in $n+r$ variables. To compute the comparative statics, we differentiate this system with respect to θ_i, yielding a Jacobian matrix that includes both the second-order partials of f and the constraint coefficients. The key object is the bordered Hessian matrix:

$${}^{r}[H] = \begin{bmatrix} \mathbf{0}_{r \times r} & G^{\top} \\ G & H \end{bmatrix}, \tag{4}$$

where G is the $r \times n$ matrix of constraint gradients ($G_{\beta j} = g_j^\beta$), and the $r \times r$ block of zeros arises because the constraints are linear.

Let ${}^rH_{ii}$ be the cofactor of the i-th diagonal element in the bordered Hessian. Then:

$$\left(\frac{\partial x_i}{\partial \theta_i}\right)_r = \frac{{}^rH_{ii}}{{}^rH}. \tag{5}$$

Under the maintained assumptions of strict concavity and regularity, this derivative remains negative:

$$\left(\frac{\partial x_i}{\partial \theta_i}\right)_r < 0.$$

Finally, by Jacobi's theorem on bordered determinants, we obtain the monotonicity result:

$$\left(\frac{\partial x_i}{\partial \theta_i}\right)_r \geq \left(\frac{\partial x_i}{\partial \theta_i}\right)_{r+1},$$

and hence the chain:

$$\left(\frac{\partial x_i}{\partial \theta_i}\right)_0 \leq \left(\frac{\partial x_i}{\partial \theta_i}\right)_1 \leq \cdots \leq \left(\frac{\partial x_i}{\partial \theta_i}\right)_r < 0. \tag{6}$$

This completes the semi-formal proof of Theorem 1.

This result is economically intuitive. Constraints reduce flexibility. In a cost minimization problem, fixed inputs in the short run (e.g., capital) prevent full reoptimization after a price change. As a result, short-run responses are muted compared to long-run ones.
This logic applies widely:

- Labor demand is more elastic in the long run.
- Consumption responds more to income when liquidity constraints are relaxed.
- Monopoly output adjusts more when capacity constraints are lifted.

2.1.2 Application: Labor Demand in the Short Run vs. Long Run

We now formalize the Le Châtelier-Samuelson prediction in the context of labor demand. Consider a profit-maximizing firm with a strictly concave production function $F(K, L)$ and output price p. The firm faces factor prices (r, w) for capital K and labor L, respectively. Profit is given by:

$$\pi(K, L; r, w, p) = pF(K, L) - rK - wL. \tag{7}$$

Let $K^0(w, r, p)$ and $L^0(w, r, p)$ denote the firm's long-run input demand functions — i.e., the solution to the unconstrained maximization of (7) over (K, L). Now suppose that in the short run, capital is fixed at $\bar{K}$ and the optimal labor demand function under this

fixed capital amount $\bar{K}$ is given by $\tilde{L}(K, w, r, p)$. Then, from a well-known identity

$$L^0(w, r, p) \equiv \tilde{L}(K^0(w, r, p), w, r, p) \tag{8}$$

which further means that the unrestricted choice for labor is the same as the restricted choice when capital is chosen at its unrestricted level, we can imply the following expressions:

$$\frac{\partial L^0}{\partial w} \equiv \frac{\partial \tilde{L}}{\partial w} + \frac{\partial \tilde{L}}{\partial K} \cdot \frac{\partial K^0}{\partial w} \tag{9}$$

$$\frac{\partial L^0}{\partial r} \equiv \frac{\partial \tilde{L}}{\partial r} + \frac{\partial \tilde{L}}{\partial K} \cdot \frac{\partial K^0}{\partial r} \tag{10}$$

Since in the short run, capital is fixed, we write. $\frac{\partial \tilde{L}}{\partial r} = 0$. Further, If we apply Envelope Theorem to the long run unconstrained maximization of (7) over (K, L), we obtain $\frac{\partial L^0}{\partial K^0} = \frac{\partial\, r}{\partial\, w}$. Using these two results, (10) simplifies to:

$$\frac{\partial L^0}{\partial r} \equiv \frac{\partial \tilde{L}}{\partial K} \cdot \frac{\partial K^0}{\partial r} \equiv \frac{\partial K^0}{\partial w} \tag{11}$$

Now, substituting the expression for $\frac{\partial K^0}{\partial w}$ from (11) into (9) gives us the Le Châtelier prediction:

$$\frac{\partial L^0}{\partial w} = \frac{\partial \tilde{L}}{\partial w} + (\frac{\partial \tilde{L}}{\partial K})^2 \cdot \frac{\partial K^0}{\partial r} < \frac{\partial \tilde{L}}{\partial w} < 0 \tag{12}$$

This is true having known that the factor demand curves have a downward slope for both the short and long run.

This result formalizes the intuitive idea that labor demand is more elastic in the long run than in the short run. When capital is fixed, the firm has fewer adjustment margins in response to a wage change. Once capital becomes adjustable, the firm can reoptimize fully, leading to a larger (in magnitude) response of labor to changes in w. It is also quite relevant to note here that this derivation depends only on the differentiability and concavity of the production function and does not require sign assumptions on cross-partials beyond local substitutability of labor and capital.

2.2 Limits of the Le Châtelier-Samuelson Principle

As observed in the previous section, the Le Châtelier-Samuelson principle, as classically formulated, is fundamentally a local result. It relies on the differentiability and concavity of the objective function and yields conclusions about comparative statics in response to infinitesimal changes in parameters. However, this locality means that the principle cannot be applied to discrete or finite changes without imposing additional global assumptions.

Samuelson himself acknowledged this strong restriction. In a 1965 postscript to his 1949 paper, he explicitly noted that the Le Châtelier principle "is indeed not true in the large" and constructed examples in which a finite price increase leads to a smaller

adjustment in a less constrained problem. This occurs in regions where second-order cross-partials—such as $\partial^2 f/\partial x_i \partial x_j$—change sign, violating the assumptions underpinning local monotonicity. This was further corroborated by Paul Milgrom and John Roberts in their 1996 paper in which they demonstrated several instances where the responsiveness of a decision variable, such as labor demand, can be greater when certain constraints are present, in direct contradiction to the local prediction of the Le Châtelier principle. We show one such instance as follows:

2.2.1 Example: Non-monotonic labor demand under fixed capital

Suppose a firm can produce one unit of output using either two units of labor, or one unit of labor combined with one unit of capital, or it can shut down and produce nothing. It can also engage in any convex combination of these three production plans. We represent the firm's production possibility set by the extreme points $(0, 0, 0)$ (shutdown), $(-1, -1, 1)$ (labor and capital), and $(-2, 0, 1)$ (labor-only), where each triple corresponds to (labor, capital, output).

At the initial price vector $(w, r, p) = (0.7, 0.8, 2)$, the firm maximizes profits by choosing the labor-only option $(-2, 0, 1)$, yielding a profit of $2 - 2 \cdot 0.7 = 0.6$. Now suppose the wage increases to $w = 1.1$, resulting in the new price vector $(1.1, 0.8, 2)$. Under these new prices, the labor-only option yields a negative profit of $2 - 2 \cdot 1.1 = -0.2$, and the mixed labor-capital option $(-1, -1, 1)$ yields a lower but still positive profit of $2 - 1.1 - 0.8 = 0.1$. Thus, in the long run, the firm adjusts by switching to the mixed technology, reducing its labor demand from 2 to 1.

However, if capital is fixed in the short run—say, the firm has no capital available—then the only feasible options are to either use two units of labor or shut down. Since the labor-only plan now earns a loss, the firm prefers to shut down entirely, reducing its labor demand from 2 to 0. In this case, the short-run labor demand adjustment is larger than in the long run.

This directly contradicts the local version of the Le Châtelier-Samuelson principle, which asserts that the more flexible (less constrained) system should always exhibit greater responsiveness. In contrast, this example shows that when discrete production technologies and non-differentiabilities are involved, greater constraints can lead to larger adjustments.

3 A Global Le Châtelier Principle

3.1 Feedback Effects and the Limits of Local Reasoning

Let us return to the motivating example: how should labor demand respond to a wage increase when capital is either held fixed (short run) or allowed to adjust (long run)? According to the classical Le Châtelier-Samuelson principle, the answer is unambiguous — labor demand should respond less when constraints are imposed, i.e., when capital is

fixed. But our latest example demonstrated that this conclusion can fail in the presence of discrete technologies and non-differentiabilities.

Still, one might feel that the long-run response should generally be larger. Various intuitive arguments have been offered, but perhaps the most compelling and accurate rests on the idea of positive feedbacks between input adjustments.

Suppose capital and labor are substitutes in the sense that an increase in usage of one reduces the marginal productivity of the other — a condition that also implies they are substitutes in the demand-theoretic sense (i.e., a lower price of labor reduces demand for capital). Now consider a fall in the price of labor. Undoubtedly, labor soars in the short run. In the long run however, when capital is adjustable, the firm may additionally reduce its use of capital in response to the cheaper labor. Since now inputs are substitutes, employing less capital increases the marginal productivity of labor only reinforces the incentive to hire more labor. The flexibility in capital adjustment thus generates a positive feedback on labor demand: an indirect effect that strengthens the direct one.

Alternatively, suppose capital and labor are complements. Then a fall in the wage leads to an short run increase in labor demand. In the long run, the firm may respond by increasing capital to match the larger labor force — again raising the marginal productivity of labor and encouraging further labor hiring. In both scenarios we see that capital's adjustment amplifies labor demand. The common thread is that feedback from capital reinforces the initial adjustment in labor — a dynamic absent in the short-run, where capital is held fixed.

These feedback effects do not rely on smooth technologies, infinitesimal price changes, or convexity. The mechanism works just as well when the firm must reconfigure its production technology, reorganize workflows, or make binary decisions about capacity. For instance, a real wage drop may justify investing in organizational changes that support managing a larger labor force — training, supervision, even new divisions of labor. These long-run adaptations or multiplier effects, are simply unavailable in the short-run.

This view of comparative statics, grounded in feedback logic rather than calculus, suggests that the classical Le Châtelier principle — limited to derivatives at a point — is simply too narrow. What the above examples hint at is a broader, more general principle: when variables reinforce each other, relaxing constraints should increase responsiveness — globally, not just locally.

This is the central contribution of Milgrom and Roberts (1996): they formalize a global Le Châtelier principle that applies not just to smooth optimization problems, but to systems with non-differentiable responses, discrete actions, and large parameter changes. The principle captures the amplifying role of feedbacks in adjustment processes. When inputs are uniformly substitutes or uniformly complements across the relevant domain, long-run responses exceed short-run responses — not because of curvature or convexity, but because long-run choice enables an additional layer of adjustment.

Not surprisingly, such global principle should only fail in adverse scenarios, when inputs are substitutes in some regions and complements in others, as in our earlier example where

capital demand rises and then falls as wages change. In such cases, the feedback structure changes direction midstream, and the system no longer provides consistent amplification.

In what follows, we explore this modern framework. We introduce some concepts of monotone comparative statics that underlie the generalization of the Le Châtelier principle to non-local settings. These tools will allow us to precisely characterize when — and why — relaxing constraints leads to greater responsiveness.

3.2 Supermodularity and the Architecture of Positive Feedback

To formalize the intuition behind feedback effects — that adjustments in one variable amplify or reinforce adjustments in another — we turn to structural properties of the objective function that capture complementarities in decision-making. The key mathematical tools here are *increasing differences*, *supermodularity*, and their ordinal counterparts: the *single crossing property* and *quasi-supermodularity*.

These concepts originate from lattice theory, a branch of order theory. Since they rely on ordering relationships rather than calculus or convexity, they are well-suited for analyzing non-differentiable systems, discrete choices, and environments with limited structure.

Posets and Lattices A *partially ordered set*, or *poset*, is a set X equipped with a binary relation $\leq$ that satisfies reflexivity, antisymmetry, and transitivity. Many economic choice spaces — quantities, binary decisions, or multi-dimensional configurations — naturally form posets.

If, for every pair $x, x' \in X$, there exist a least upper bound (called the *join*, denoted $x \vee x'$) and a greatest lower bound (called the *meet*, denoted $x \wedge x'$) contained in X, then X is a *lattice*. These operations generalize coordinate-wise maximum and minimum: in $\mathbb{R}^n$, $x \vee x'$ is formed by taking the max in each coordinate, and $x \wedge x'$ the min. The existence of joins and meets is critical to defining complementarities in a general setting.

Increasing Differences and the Single Crossing Property

Definition 1. *Let $f : X \times \Theta \to \mathbb{R}$ where X is a lattice and Θ is a poset. The function f has* increasing differences *in (x, θ) if:*

$$x' > x,\ \theta' > \theta \ \Rightarrow\ f(x', \theta') - f(x, \theta') \geq f(x', \theta) - f(x, \theta).$$

This means that the marginal benefit of increasing x is larger when θ is higher — that is, x and θ are *complements*. In a production setting, this captures the idea that the value of labor increases with more capital, and vice versa.

Increasing differences imply strong monotone comparative statics: if f has increasing differences and $x^*(\theta)$ maximizes $f(x, \theta)$, then $x^*(\theta)$ is non-decreasing in θ.

The *ordinal* analogue of increasing differences is the *Single Crossing Property* (SCP):

Definition 2 (Single Crossing Property)**.** *A function $f(x, \theta)$ satisfies the* single crossing property *in (x, θ) if for any $x' > x$, the sign of $f(x', \theta) - f(x, \theta)$ is non-decreasing in θ.*

This means that once x' becomes preferable to x at some θ, it remains so at all higher values of θ. SCP captures comparative statics behavior based on ordinal preferences, and is especially useful in non-smooth or discrete settings.

Supermodularity and Quasi-Supermodularity

Definition 3 (Supermodular functions, Milgrom and Roberts (1990))**.** *Consider a function $f : X \to \mathbb{R}$ where X is a lattice. The function is* supermodular *if for all $x, x' \in X$:*

$$f(x \vee x') + f(x \wedge x') \geq f(x) + f(x').$$

This condition captures *positive feedback* or *complementarity* between variables: increasing one variable raises the marginal return to increasing another. In differentiable settings, this corresponds to non-negative cross-partial derivatives:

$$\frac{\partial^2 f}{\partial x_i \partial x_j} \geq 0.$$

Supermodularity ensures that optimal decisions move in the same direction in response to external changes, a signature of systems with reinforcing mechanisms.

The *ordinal* counterpart to supermodularity is *quasi-supermodularity*:

Definition 4 (Quasi-Supermodularity, Milgrom and Roberts (1990))**.** *A function f is* quasi-supermodular *on a lattice X if for all $x, x' \in X$:*

- *If $f(x') > f(x)$, then $f(x \vee x') \geq f(x)$ and $f(x \wedge x') \leq f(x')$.*

This property ensures that if x' is preferred to x, then moving from x toward x' via either coordinate-wise increase (join) or decrease (meet) preserves or strengthens this preference. While weaker than full supermodularity, quasi-supermodularity is still sufficient for many monotone comparative statics results, especially when paired with SCP.
Finally, another powerful tool that addresses comparative statics analysis in settings where the objective function exhibits complementarities in form of increasing differences or supermodularity is the Topkis' Theorem. Specifically, Topkis' Theorem establishes conditions under which a decision-maker's optimal choices move monotonically with changes in a parameter, even when the problem involves non-differentiable or discrete choices.

Theorem 2 (Theorem, Topkis (1978))**.** *Let X be a lattice and Θ a totally ordered set. Suppose $f : X \times \Theta \to \mathbb{R}$ satisfies the following:*

1. *f has increasing differences in (x, θ).*
2. *For each $\theta \in \Theta$, the feasible set $X(\theta) \subseteq X$ is a sublattice and satisfies $X(\theta) \subseteq X(\theta')$ whenever $\theta \leq \theta'$.*

Then, the set of maximizers

$$x^*(\theta) \in \arg\max_{x \in X(\theta)} f(x, \theta)$$

is non-decreasing in θ in the strong set order. That is, for $\theta' > \theta$, the minimal element of $x^*(\theta')$ is at least as large as the maximal element of $x^*(\theta)$.

When the objective exhibits increasing differences in the choice variable and parameter, and when the feasible sets expand with the parameter, the optimal decision rule is monotone: higher parameters lead to higher optimal choices. This applies even in settings with discrete actions or non-smooth payoffs.
Above concepts extend the logic of the classical Le Châtelier principle. They clarify when allowing more flexibility — relaxing a constraint, expanding the decision set — amplifies the system's responsiveness. Crucially, they do so without relying on smoothness or convexity, but instead on the *structure of complementarities and order*.
In the next section, we apply these ideas to characterize comparative statics in a global setting, using monotone methods to generalize responsiveness across large parameter changes and discrete technologies.

3.3 Extending Local Samuelson Insights

The previous section introduced the concepts of supermodularity and increasing differences, emphasizing their role in generating monotone and reinforcing responses to parameter changes. These ideas now culminate in a global version of the Le Châtelier principle — one that formalizes when relaxing a constraint leads to a larger adjustment, even for finite changes or discrete decision spaces.

Setup For achieving coherence and maintaining comparability with the local Le Châtelier-Samuelson results studied earlier, let us consider again our example of a profit-maximizing firm from Section 2.1.2. The firm has a production technology $f(x, y)$ and faces output price p and factor prices (r, θ) for capital $y \in Y$ and labor $x \in X$, respectively. To generalize beyond the wage-centric framing, we make a simple substitution: $\theta = -w$. The firm's objective is to maximize profits given by:

$$F(x, y, \theta) = pf(x, y) - ry + \theta x.$$

We treat p and r as fixed throughout and focus on how labor demand responds to changes in θ. Assuming X and Y to be two compact sub-lattices of $\mathbb{R}$ such that F is order upper semi- continuous in (x, y), we say by Tarski's theorem (Tarski (1955); Etessami et al. (2019)) that there exists at least one maximum in each problem. Define:

$$x^*(\theta, y) \in \arg\max_{x \in X} F(x, y, \theta)$$

$$y^*(\theta) = \arg\max_{y \in Y} F(x^*(\theta, y), y, \theta)$$

Note: In case of multiple maximizers in either problem, we consider the largest one.
In the short run, capital is held fixed — but crucially, this fixed capital level depends on a past wage θ_1. Therefore, short-run labor demand depends on both θ_1 (previous wage) and θ_2 (current wage), and is denoted by $x^*(\theta_2, y^*(\theta_1))$. In contrast, long-run labor demand $x^*(\theta_2, y^*(\theta_2))$ reflects full adjustment.

3.4 Global Le Châtelier Principle: Supermodular Case

Theorem 3 (Milgrom and Roberts, 1996)**.** *Suppose $F(x, y, \theta)$ is supermodular. Then for any $\theta_1 \geq \theta_2$,*

$$x^*(\theta_2, y^*(\theta_2)) \leq x^*(\theta_2, y^*(\theta_1)) \leq x^*(\theta_1, y^*(\theta_1)).$$

Proof. Given $F(x, y, \theta)$ is supermodular if the cross partial derivatives $F_{xy}, F_{x\theta}$ and $F_{y\theta}$are non negative at all points in the domain, signifying complementarities between choice variables as well as between the choice variables and the parameter. Then by Topkis's Theorem (1978), $y^*(\cdot)$ and $x^*(\cdot, \cdot)$ are monotonically non decreasing. Hence, $y^*(\theta_2) \leq y^*(\theta_1)$ and $x^*(\theta_2, y^*(\theta_2)) \leq x^*(\theta_2, y^*(\theta_1)) \leq x^*(\theta_1, y^*(\theta_1))$. □

An important note here is that the assumption of supermodularity in F and more precisely, $F_{xy} \geq 0$ implicitly assumes that $f_{xy} \geq 0$, further implying that x and y are complements. Hence this result only ensures that increases in y raise the marginal value of x, and the comparative statics of x and y with respect to θ are monotone. Consequently, adjusting y introduces a feedback that amplifies the system's response in x.
Note: Theorem 3 also includes the case where x and y are substitutes everywhere, i.e $f_{xy} \leq 0$. When $f(x, y)$ is submodular, $F(x, y, \theta)$ in the way that it is defined, continues to be supermodular and Theorem 3 applies.

Complements and Substitutes Theorem 3 subsumes both the complements and substitutes cases. If x and y are substitutes, then an increase in θ (a wage cut) leads to more labor but less capital, which — through negative feedback — still reinforces labor demand. If x and y are complements, both increase together, reinforcing labor demand via positive feedback. In either case, the feedback loop is consistent, and monotonicity guarantees that relaxing the constraint leads to a larger response. The global principle provides a rigorous foundation for the intuition that long-run responses exceed short-run ones due to reinforcing feedbacks.
Importantly, this framework avoids assumptions about differentiability, convexity, or continuity in choice sets. The results hold equally in problems with discrete actions, binary constraints, or threshold effects — so long as the relevant monotonicity and complementarity conditions hold throughout the domain.

Limits and Failures. The principle can fail when inputs switch between being substitutes and complements as parameters change. In such cases, the direction of feedback may reverse, breaking the required monotonicity. The presence of such "non-uniform" complementarity undermines global amplification and creates room for counterexamples such as the one we described in Section 2.2.1.

3.5 Weak Form Global Principle

The requirement of global supermodularity (or uniform complementarity/substitutability) can be unnecessarily strong. Fortunately, a powerful generalization due to Milgrom and Shannon (1994) shows that one can replace supermodularity with its ordinal counterpart — quasi-supermodularity — and replace increasing differences with the single crossing property (SCP). Additionally, there is no reason to restrict ourselves to the study of optimization problems with only two real variables. We would essentially expect x and y to take values in any lattice, including $\mathbb{R}^n$ and $\mathbb{R}^m$ provided X and Y are restricted to be sub-lattices. The following theorem from Milgrom and Roberts (1996) comes as a more general version of Theorem 3 but one that does away with both of the limitations outlined earlier.

Theorem 4 (Milgrom and Roberts (1996))**.** *Let X and Y be lattices and S be the sub-lattice of the product lattice $X \times Y$. Let $(x^*(\theta), y^*(\theta))$ be the least upper bound of* $\arg\max_{(x,y)\in S} F(x, y, \theta)$ *and $x^*(\theta, y)$ be the least upper bound of* $\arg\max_{x\in\{x|(x,y)\in S\}} F(x, y, \theta)$*. Suppose $F(x, y, \theta)$ is continuous in the order-interval topology, quasi-supermodular in (x, y) and has the single crossing property in $(x, y; \theta)$ Then for any $\theta_1 \geq \theta_2$,*

$$x^*(\theta_2, y^*(\theta_2)) \leq x^*(\theta_2, y^*(\theta_1)) \leq x^*(\theta_1, y^*(\theta_1)).$$

This version of the global Le Châtelier principle thus applies in more general settings — including non-smooth, discrete, or threshold-based technologies — where only the ordinal structure of feedback and monotonicity is preserved.

3.6 Jamison's Lattice-Theoretic Generalization

Building on the ordinal framework of Milgrom and Shannon (1994), Jamison (2006) establishes an even weaker version of the global result given in Theorem 4. In addition to allowing for non-differentiable objective functions which satisfy quasi-supermodularity and single crossing, and work with discrete actions, he also accounts for changes in the choice set itself as the parameter varies. The decision variables are indexed as $(x_1, \ldots, x_n)$, each drawn from a lattice, and the parameters (e.g. prices) from a partially ordered set. Jamison's key contribution is a generalized comparative statics theorem that shows: *as more variables are released from constraint and allowed to reoptimize in response to a parameter shift, their marginal responsiveness increases monotonically.* Formally, if $x_i^{(r)}$ denotes the optimal choice of variable x_i when only the last $n-r$ variables are re-optimized

(holding the first r fixed at their previous optimal values), then for all i and for any upward change in the parameter θ, the following holds:

$$x_i^{(0)}(\theta') \geq_s x_i^{(1)}(\theta, \theta') \geq_s \cdots \geq_s x_i^{(n)}(\theta, \theta') \geq_s x_i^{(0)}(\theta),$$

where $\geq_s$ denotes the *Viennot (1986)'s strong set order* used in lattice theory. That is, the more variables are allowed to adjust freely (moving from short-run to long-run), the larger the induced change in any given choice variable.

This nesting result is a formal restatement of the Le Châtelier intuition: increased flexibility leads to stronger feedback effects and thus to greater responsiveness. Compared to Milgrom and Roberts (1996), Jamison's formulation accommodates variable choice domains and avoids requiring differentiability or even cardinality, relying solely on ordinal structure (quasi-supermodularity and the single crossing property). His theorem thus affirms that the Le Châtelier logic extends robustly to abstract, discrete, and combinatorial environments.

All of the aforementioned results capture the structural logic behind the Le Châtelier effect: allowing an additional variable to respond opens up feedback loops that amplify the response of interest. Whether through the strong lens of supermodularity or the weaker but robust lens of quasi-supermodularity and SCP, the conclusion is the same: *greater flexibility in adjustment leads to greater responsiveness, provided the feedback structure is monotone and aligned.*

4 Comparative Statics Under Uncertainty

In the preceding sections, we have shown — both locally through Samuelson's bordered Hessian analysis and globally via lattice-theoretic tools — that labor demand is more elastic in the long run than in the short run. This fundamental prediction of the Le Châtelier principle hinges on the idea that relaxing constraints (e.g., making capital adjustable) introduces reinforcing feedbacks that amplify a system's response.

However, all of our analysis so far has assumed a deterministic environment. In real-world settings, firms often make decisions under uncertainty — particularly regarding output prices, input costs, and policy shocks. A natural question thus arises: *how does uncertainty affect the structure of short-run versus long-run comparative statics?* Does the Le Châtelier principle still apply when outcomes are stochastic? How do risk preferences shape these dynamics?

In this section, we extend the framework to incorporate uncertainty and risk aversion. We begin by surveying key contributions in the literature that address comparative statics under uncertainty, including foundational results by Sandmo (1971), Batra and Ullah (1974), Nocetti (2016) and Diamond and Stiglitz (1973). These works primarily focus on

static environments where firms maximize expected profits or expected utility, and they identify conditions under which output and input demands are monotonic in risk or in the variability of uncertain prices.

We then define two formal models — one in which the firm is risk-neutral and maximizes expected profits under stochastic output prices, and another in which the firm is risk-averse and maximizes expected utility. The first model follows the approach of Hennessy (1997), who provides short-run and long-run comparative statics under output price uncertainty. The second builds on the analysis of Snow (2000), who extends the Le Châtelier principle to a risk-averse firm facing output price uncertainty.
These models help us understand two central questions:

1. Does the classical Le Châtelier prediction — that long-run responses dominate short-run ones — continue to hold under uncertainty?

2. How do risk preferences interact with adjustment frictions to shape comparative statics outcomes?

The broader objective is to investigate whether a generalization of the Le Châtelier principle survives in settings where firms must optimize under both constraints and risk. We will ultimately use these insights to motivate a dynamic formulation in the next section, where firms make sequential decisions in the face of both adjustment costs and evolving uncertainty.

4.1 Comparative Statics of a Risk-neutral Firm facing Output Price Uncertainty

We now turn to the work of Hennessy (1997), who rigorously studies the comparative statics of input demand under uncertainty of output prices, specifically in the context of *risk-neutral firm*. His analysis builds on the classical Le Châtelier logic, but adapts it to stochastic environments with price risk, showing how both the nature of the shock and the form of production interact to shape responsiveness.

Let the firm face a strictly concave, twice differentiable production function $f(K, L)$, where L represents labor (short-run flexible input) and K represents capital (long-run flexible input). The output price $\tilde{p}$ is a random variable supported on $[\underline{p}, \overline{p}]$ with cumulative distribution $G(\tilde{p}; \lambda, \Upsilon)$, parameterized such that:

- increases in λ correspond to first-order stochastic dominance (FOSD) shift:

$$\lambda_2 \geq \lambda_1 \implies G(\tilde{p}; \lambda_2, \Upsilon) \leq G(\tilde{p}; \lambda_1, \Upsilon)$$

- increases in Υ correspond to mean-preserving contractions (MPC):

$$\Upsilon_2 \geq \Upsilon_1 \implies \int_{\underline{p}}^{\nu} G(\tilde{p}; \lambda, \Upsilon_2)\, d\tilde{p} \leq \int_{\underline{p}}^{\nu} G(\tilde{p}; \lambda, \Upsilon_1)\, d\tilde{p} \text{ for all } \nu \in [\underline{p}, \bar{p}]$$

The firm is risk-neutral and chooses inputs to maximize expected profits before the random $\tilde{p}$ is realized.

$$\max_{K,L} \int_{\underline{p}}^{\overline{p}} \Pi(K, L; \tilde{p}) \, dG(\tilde{p}; \lambda, \Upsilon).$$

where $\Pi(K, L; \tilde{p})$ represents the profits to the firms. Such decision environment could be used to mimic the case of an educational establishment facing random per day demand for services, makes long run investment decisions (like investing in the construction of a new building) but only short run labor decisions (recruiting academics, hiring local staff etc.); or a horticulturist who faces a year by year random production and makes long run capitalization decisions but only short run fertilization decisions.
Hennessy (1997) derives three central results about the responsiveness of labor and capital under this stochastic framework.

Lemma 1 (FOSD Shifts; Hennessy (1997))**.** *If $\Pi(K, L; \tilde{p})$ is supermodular in (K, L) and the output price distribution shifts via first-order stochastic dominance (FOSD), then (a) Labor demand L increases in the short run (b) Both capital K and labor L increase in the long run (c) The long-run increase in L exceeds the short-run increase.*

Lemma 2 (Mean Preserving Contraction, Hennessy (1997))**.** *If the profit function satisfies cross-partial positivity ($\Pi_{KL} \geq 0$) and the marginal effect of the shock on profit is decreasing, i.e $\partial^3\Pi(K, L, \tilde{p})/\partial K \, \partial\tilde{p}^2$ and $\partial^3\Pi(K, L, \tilde{p})/\partial L \, \partial\tilde{p}^2$ then a mean-preserving contraction (MPC) of the price distribution leads to Labor increases in the short run, Capital and Labor both increase in the long run, Long-run labor demand increases more than in the short run.*

Second Order Stochastic Dominance partially orders price distributions according to their impact on the expected value of all increasing concave utility functions.

Lemma 3 (SOSD shifts; Hadar and Seo (1990); Baccelli and Makowski (1989))**.** *If the profit function $\Pi(K, L, \tilde{p})$ is supermodular and the marginal effect of the shock decreases in the shock variable i.e $\partial^3\Pi(K, L, \tilde{p})/\partial K \, \partial\tilde{p}^2$ and $\partial^3\Pi(K, L, \tilde{p})/\partial L \, \partial\tilde{p}^2$, then under a second-order stochastic dominance (SSD) shift in $\tilde{p}$ (a) Labor increases in the short run (b) Capital and labor both increase in the long run (c) Labor's long-run increase exceeds its short-run increase.*

Together, these results show that the Le Châtelier logic persists even under stochastic output prices: allowing more inputs to adjust amplifies the firm's responsiveness to environmental changes. Hennessy's analysis is notable for relying on relatively weak assumptions—mainly concavity and supermodularity—and thus provides a robust benchmark for assessing comparative statics under uncertainty.
Note that the given functional form for profit $\Pi(K, L; \tilde{p})$ could also be non linear in random price $\tilde{p}$. However, in all instances of linearity, such as when $\Pi(K, L; \tilde{p}) = \tilde{p}F(K, L) - rK - wL$, we find that the aforementioned results tend to be quite trivial.

4.1.1 Extensions

Milgrom and Shannon (1994) have shown that a functional structure more general than supermodularity (quasisupermodularity in the choice variables), together with the single crossing property for changes in the exogenous variable, may often be sufficient to sign comparative statics, and that an arbitrary number of endogenous and exogenous arguments can be accommodated. Milgrom and Roberts (1996) have shown that their Le Châtelier result is robust to such a generalization when there are two endogenous arguments and a single exogenous argument, so it may be that the structure required on reward functions to give the Le Châtelier effect for certain stochastic shifts can also be relaxed. However, Athey and Schmutzler (1995) has studied related structures for monotonicity of choice in stochastic optimization problems, and her results suggest limitations on relaxing the assumptions made in Lemmas 1,2,3.

4.2 Risk-Averse Firm Under Input Price Uncertainty

We now consider the case where the firm is risk-averse and maximizes the expected utility of profit. This introduces an important new element: risk preferences now interact with technological adjustment, altering the comparative statics structure.
We follow the analysis of Snow (2000), who generalizes the Le Châtelier principle to a risk-averse firm facing output price uncertainty. The key insight is that while output supply continues to obey a generalized Le Châtelier prediction, labor demand need not—unless risk preferences take a specific form.
Let the firm maximize:

$$\max_{K,L} \; V(K, L, w, r, \varphi) = \int_{\underline{p}}^{\overline{p}} U(\Pi(K, L; \tilde{p}))\; dH(\tilde{p}; \varphi). \tag{13}$$

where U is a strictly increasing, concave utility function, $\Pi(K, L; \tilde{p}) = \tilde{p}F(K, L) - rK - wL$, and H is the cumulative distribution of the stochastic output price, parameterized by φ (which we discuss in the next section). We hypothesize that a long run utility maximum exists with positive profit for every realization of $\tilde{p}$. Therefore, the structure of comparative statics now depends critically on how risk aversion interacts with adjustment constraints.
In the certainty case—or when the firm is risk-neutral—comparative statics reflect only the curvature of technology and the relaxation of constraints. But once we introduce risk aversion, an additional force emerges: wealth effects. A shock (e.g., a wage increase) not only changes marginal costs but also alters expected profit, which in turn may change the firm's willingness to bear risk. The direction and strength of this effect depend on the form of risk preferences. In what follows, we examine how the Le Châtelier principle behaves under three canonical types of risk aversion: CARA, DARA, and IARA.

4.2.1 Constant Absolute Risk Aversion (CARA)

The case of constant absolute risk aversion offers a clean benchmark. Under CARA, the firm's utility function takes the exponential form:

$$U(\Pi) = -\exp(-\rho\,\Pi),$$

where ρ is the coefficient of absolute risk aversion. Crucially, CARA preferences imply that the firm's risk aversion is invariant to changes in wealth.

This property has powerful consequences. From Snow (2000), the long-run wage effect on labor demand can be expressed as:

$$\frac{\partial L^*}{\partial w} = \Theta\frac{\partial \tilde{L}}{\partial w} - \Theta\frac{V_{KL}}{V_{LL}V_{KK}}\,L\int U''(pF_K - r)\,dH(p;\varphi), \tag{14}$$

with $\Theta = \frac{V_{KK}\,V_{LL}}{V_{KK}\,V_{LL}-V_{KL}^2} > 1$. Under CARA, the integral vanishes since U''/U' is constant, eliminating wealth effects. Thus, only the first (direct) term matters, ensuring:

$$\frac{\partial L^{LR}}{\partial w} \;\leq\; \frac{\partial L^{SR}}{\partial w} < 0.$$

Hence, the Le Châtelier prediction holds exactly as in the certainty case. Output supply and capital demand similarly adjust more in the long run than in the short run.

4.2.2 Decreasing Absolute Risk Aversion (DARA)

Under DARA, risk aversion declines with wealth:

$$\frac{d\rho(\Pi)}{d\Pi} < 0.$$

An example is the CRRA utility function,

$$U(\Pi) = \frac{\Pi^{1-\gamma}}{1-\gamma}, \quad \gamma > 0,$$

which implies DARA when $\gamma < 1$.

Returning to (14), the wealth effect no longer vanishes. Snow decomposes the cross-partial as:

$$V_{KL} = \left(\tfrac{w}{r}\right)\int U''(pF_K - r)\,(pF_K - r)^2\,dH(p;\varphi) + F_{KL}\int U'(pF_K - r)\,p\,dH(p;\varphi). \tag{15}$$

Under DARA, the first integral amplifies the negative response to a wage increase: lower profits make the firm more risk-averse, discouraging risky production plans. This strengthens the decline in labor demand beyond the direct marginal cost effect.

Whether the long-run response obeys the Le Châtelier inequality depends on the sign of F_{KL} in (15). If capital and labor are complements, the effect is magnified and the inequality holds. If they are substitutes, wealth effects can dominate and the prediction may fail.

4.2.3 Increasing Absolute Risk Aversion (IARA)

With IARA, risk aversion rises with wealth:

$$\frac{d\rho(\Pi)}{d\Pi} > 0.$$

Now, a wage increase reduces profit and thereby *reduces* risk aversion. Returning again to (14), the wealth-effect term can offset the direct effect of higher marginal cost.

Specifically, in (15), the U'' term may now reverse sign: the induced fall in risk aversion encourages the firm to take on more risk, potentially increasing input use even as wages rise. Thus, long-run labor demand may decline less than in the short run, or even increase:

$$\frac{\partial L^{LR}}{\partial w} > \frac{\partial L^{SR}}{\partial w},$$

violating the Le Châtelier prediction. As Snow (2000) shows, such violations can occur even with standard functional forms, such as Cobb–Douglas production under CRRA preferences.

In summary, while CARA ensures that Le Châtelier predictions hold and DARA often reinforces them, IARA can generate counterintuitive outcomes in which long-run adjustments move against the standard principle. This highlights the importance of understanding not just technological curvature but also the stochastic structure of prices. In the next section, we turn from risk-preference assumptions to properties of the price distribution itself, focusing on monotone likelihood ratio (MLR) improvements as a powerful comparative statics tool.

4.3 Monotone Likelihood Ratio (MLR) Improvements in Price Uncertainty

A separate and elegant approach to modeling uncertainty is through Monotone Likelihood Ratio (MLR) improvements in the output price distribution. These are structured shifts where the distribution tilts toward higher prices in a likelihood ratio sense. Formally,

Definition 5 (Monotone Likelihood Ratio (MLR) Shifts; Karlin and Rubin (1956))**.** *A MLR shift from initial price distribution $H_1(\tilde{p};\varphi)$ to final price distribution $H_2(\tilde{p};\varphi)$ is a first order stochastic dominance shift which satisfies the following restrictions:*

$$H_1(\tilde{p};\varphi) = 1 \text{ over a nondegenerate interval } (p_2, \bar{p}]$$

$$H_2(\tilde{p};\varphi) = 0 \text{ over a nondegenerate interval } [\underline{p}, p_1)$$

And the ratio of likelihoods,

$$\frac{dH_2(\tilde{p};\varphi)}{dH_1(\tilde{p};\varphi)} = k(p) \text{ is increasing with respect to } \tilde{p} \text{ with } k \geq 0 \text{ and } k' > 0$$

As a consequence of which expected value of $\tilde{p}$ increases.

Intuitively, MLR improvements in the distribution of $\tilde{p}$ correspond to the possibility of a higher ex-post market strength of price realizations.

4.3.1 Why Focus on MLR Improvements?

The emphasis on monotone likelihood ratio (MLR) improvements is motivated by two fundamental considerations.

First, prior research has established that neither simple translations of the price distribution nor weaker stochastic shifts suffice to generate monotonic responses under risk aversion. Classic contributions by Sandmo (1971), Batra and Ullah (1974), Hoel and Vislie (1983), and Epstein (1987) demonstrate that a mere upward shift in the mean of the price distribution does not necessarily induce a higher supply response from a risk-averse firm. Similarly, subsequent work by Ormiston (1992), Ormiston and Schlee (1993), and Hadar and Russell (1978) shows that first-order stochastic dominance (FOSD) shifts are often too weak to guarantee monotonic comparative statics under risk aversion.

In contrast, MLR improvements stand out because they are *behaviorally equivalent to generalized price increases* (see Ormiston and Schlee, 1993). Such improvements tilt the likelihood ratio in favor of higher prices, ensuring that even under risk aversion, output supplied responds monotonically in the expected direction.

Second, MLR shifts are analytically attractive because they deliver Le Châtelier-type predictions without imposing restrictions on risk preferences. Unlike comparative statics under FOSD or mean-preserving shifts, where specific assumptions about risk attitudes are necessary, MLR improvements guarantee robust results across the entire class of concave utility functions. This property makes them uniquely powerful in deriving predictions about output supply under uncertainty.

Taken together, these features explain why MLR improvements merit detailed attention: they not only align with the intuition of price increases but also allow comparative statics results to be derived in a risk-preference–free manner, thereby strengthening the generality of the Le Châtelier principle under uncertainty.

Looking ahead, these insights on MLR improvements provide a natural foundation for the dynamic framework developed in Section 5, where we investigate how uncertainty and adjustment costs jointly shape the strength of Le Châtelier-type predictions over time. Formally, we summarise Snow (2000)'s findings in the following result.

Lemma 4. *Let an increase in φ imply a monotone likelihood ratio improvement in the*

distribution of $\tilde{p}$, i.e $H(\tilde{p};\varphi)$. Then irrespective of the firm's risk preferences, an increase in φ implies (a) Output increases with increase in both the short run and the long run;(b) The long-run demand for capital increases (decreases) provided capital is normal(inferior) in production; (c) Further, in the long run equilibrium under price uncertainty, output supplied increases by more in the long run than in the short run.

4.4 Increases in Risk Aversion

Finally, Snow (2000) analyzes how comparative statics behave when the firm becomes more risk-averse—modeled as a transformation in the utility function $U(\Pi;\rho)$ where ρ indexes the degree of risk aversion. Applying the results of Diamond and Stiglitz (1973), Snow establishes the following result:

Lemma 5. *For a risk averse firm which maximizes expected utility $\mathbb{E}[U(\Pi(K,L;\tilde{p}))]$, let an increase in ρ represent an increase in risk aversion. Then (a) output supplied under price uncertainty decreases when ρ increases in both the long run and the short run (b) Long run demand for capital decreases (increases) if capital is normal (inferior) in production (c) in the long run equilibrium under price uncertainty, output supplied decreases by more in the long run than in the short run.*

This confirms that the Le Châtelier principle also applies to changes in preferences themselves—at least with respect to output supply—even under general utility functions. Importantly, this result also generalizes the principle to stochastic environments with non-linear utility. Unlike the risk-neutral case, here uncertainty interacts with preferences, and comparative statics must account for the curvature of U. Snow's result thus provides a powerful validation of the Le Châtelier logic under risk aversion.

4.5 Discussion

These two models — one risk-neutral, one risk-averse — show that the Le Châtelier principle continues to hold under uncertainty, provided the firm's objective satisfies natural concavity assumptions. Whether maximizing expected profits or expected utility, the key comparative statics mechanism remains: additional flexibility leads to stronger responses.

At the same time, these results remain largely static. The firm either chooses labor once for all states (short run), or flexibly after the realization of shocks (long run). Adjustment costs, inertia, and sequential decision-making are absent.

In the next section, we close this gap. Building on the dynamic adjustment framework of Dekel et al. (2025), we study firms that reoptimize over time in the face of both stochastic shocks and frictions. Our aim is to identify dynamic analogues of the Le Châtelier principle when risk, adjustment, and timing jointly shape economic behavior.

5 A Strong Le Châtelier Principle under Uncertainty and Adjustment Costs

The classical Le Châtelier principle — in both its deterministic and stochastic forms — characterizes how optimal responses vary when constraints are relaxed. In static environments, relaxing constraints (e.g., allowing more inputs to adjust) typically leads to larger responses to shocks. Similarly, under uncertainty, we saw that optimistic shifts in beliefs (in the MLR sense) tend to increase optimal actions, especially when the decision maker is risk-neutral or exhibits certain risk preferences (e.g., CARA).

However, real-world decision problems often unfold dynamically. Firms and other economic agents face not only uncertainty but also intertemporal adjustment costs that prevent full immediate optimization. This friction introduces path dependence: agents may wish to move toward a new optimum but are discouraged from doing so abruptly due to adjustment penalties.

We now develop a dynamic model in which a decision maker (DM) repeatedly chooses actions in response to an uncertain environment, but faces per-period adjustment costs. The uncertainty is *stationary* — that is, in every period, the DM faces the same known probability distribution G over an exogenous state θ. The environment is thus stochastic but *time-invariant*. Each period, a realization of θ is drawn i.i.d. from G, and the DM chooses x_t knowing that the same distribution will govern future periods. There is no learning or updating across time.

This setup is closely aligned with the deterministic dynamic framework of Dekel et al. (2025), who show that under suitable conditions — notably, supermodularity, single-crossing, and separable adjustment costs — the optimal sequence of actions can be chosen to be non-decreasing over time. Their result relies on a method of *monotonization*, which constructs a monotone path that weakly dominates any non-monotone candidate path in value.

In our case, we extend their framework to incorporate expected utility over a fixed stochastic environment. The key observation is that, under mild regularity conditions, the expected utility function $\mathbb{E}_{\theta\sim G}[u(F(x,\theta))]$ inherits the relevant structure (supermodularity, increasing differences) from the underlying deterministic function $F(x,\theta)$. This allows us to extend Dekel et al.'s result to a stochastic setting, yielding a dynamic Le Châtelier principle under uncertainty. We now formally present the model.

5.1 Comparative Benchmark: Long-Run Frictionless Optimum

Let $G^1, G^2 \in \mathcal{D}([\underline{\theta}, \bar{\theta}])$ be two belief distributions over the random parameter $\tilde{\theta}$, with $G^2 \succ_{MLR} G^1$ in the monotone likelihood ratio (MLR) order. These represent two different belief environments having support on $[\underline{\theta}, \bar{\theta}]$, with G^2 corresponding to a more optimistic distribution over the state.

We first characterize the frictionless (static) benchmark problem. In the absence of adjustment frictions, the risk averse decision maker (DM) selects a static optimal action

under each belief:

$$\bar{x} \in \arg\max_{x \in L} \int_{\underline{\theta}}^{\bar{\theta}} u(F(x,\theta))\, dG^2(\theta) \equiv \arg\max_{x \in L} V(x, G^2),$$

$$\underline{x} \in \arg\max_{x \in L} \int_{\underline{\theta}}^{\bar{\theta}} u(F(x,\theta))\, dG^1(\theta) \equiv \arg\max_{x \in L} V(x, G^1).$$

We refer to $\underline{x}$ and $\bar{x}$ as the *frictionless benchmarks* under G^1 and G^2, respectively. Under strict quasi-concavity of $V(x, G)$ in x (uniformly across G), both optimizers are unique.

Let $x^*(\delta_\theta)$ denote the optimal action under certainty, where δ_θ is the Dirac measure that places all mass at a single θ. The necessary and sufficient first- and second-order conditions for optimality under G^1 are:

$$V_x(\underline{x}, G^1) = \int_{\underline{\theta}}^{\bar{\theta}} u_x(\underline{x}, \theta)\, dG^1(\theta) = 0, \tag{16}$$

$$V_{xx}(\underline{x}, G^1) = \int_{\underline{\theta}}^{\bar{\theta}} u_{xx}(\underline{x}, \theta)\, dG^1(\theta) < 0. \tag{17}$$

Lemma 6 (Landsberger and Meilijson (1990))**.** *For distributions $G^1, G^2 \in \mathcal{D}([\underline{\theta}, \bar{\theta}])$, we have $G^2 \succ_{MLR} G^1$ if and only if there exist points $\theta_1 \leq \theta_2$ in $[\underline{\theta}, \bar{\theta}]$ and a non-negative, increasing function $h : [\theta_1, \theta_2] \to \mathbb{R}$ such that:*

$$\int_{\underline{\theta}}^{\theta_1} dG^2(\theta) = \int_{\theta_2}^{\bar{\theta}} dG^1(\theta) = 0, \tag{18}$$

$$G^2(\theta) = \int_{\theta_1}^{\theta} h(\theta)\, dG^1(\theta), \quad \textit{for all } \theta \in [\theta_1, \theta_2]. \tag{19}$$

5.2 Dynamic Problem and Main Results

We now turn to the full dynamic problem with adjustment costs. The DM chooses an infinite sequence of actions $\{x_t\}_{t=1}^{\infty}$ over time, where each action incurs a cost for deviating from the previous period's choice which further allows for some disutility $v(\cdot)$ It must be noted that while the DM must make his decision about the choice variables under uncertainty parameterized by $\tilde{\theta}$, his realized per-period payoffs depend solely on θ realizations every period. Further, they discount future payoffs at factor $\delta \in (0, 1]$.

Definition 6 (Dynamic Program)**.** *Given an initial action $x_0 \in L$, the DM's lifetime objective is*

$$\mathcal{J}(\{x_t\}_{t=1}^{\infty}; x_0, G) := \sum_{t=1}^{\infty} \delta^{t-1} \int_{\underline{\theta}}^{\bar{\theta}} u\big(F(x_t,\theta)\big) - v\big(C_t(x_t - x_{t-1})\big)\big)\, dG(\theta),$$

where $F(x_t, \theta)$ is the gross payoff in period t, $C_t(x_t - x_{t-1})$ is the adjustment cost for

moving from x_{t-1} to x_t, $u(\cdot)$ is the period utility function applied to the deterministic payoff F, $v(\cdot)$ is the period dis-utility function applied to the adjustment costs, G is the belief distribution over θ, $\delta \in (0,1]$ is the discount factor. The DM chooses $\{x_t\}_{t=1}^{\infty}$ to maximize $\mathcal{J}$.

Per-Period Problem. In sequential-choice form, the decision in period t solves:

$$J_t(x; x_{t-1}, G) := \int_{\underline{\theta}}^{\bar{\theta}} u\big(F(x,\theta)\big) - v\big(C_t(x - x_{t-1})\big)\, dG(\theta), \tag{20}$$

where x_{t-1} is given from the previous period. The optimal choice at t satisfies:

$$x_t \in \arg\max_{x \in L} J_t(x; x_{t-1}, G).$$

Throughout our main results, we fix $G = G^2$ (the optimistic belief) and write $J_t(x; x_{t-1})$ for short.

Goal. We aim to characterize the structure of the optimal adjustment path $\{x_t\}_{t\geq 1}$, showing that it remains bounded between the static benchmarks $\underline{x}$ and $\bar{x}$ defined in the frictionless problem, and — under stronger conditions — is also monotone over time.
We adopt the following assumptions:

(A1) L is a sublattice of $\mathbb{R}^n$.

(A2) The utility function $u : \mathbb{R} \to \mathbb{R}$ is continuous, strictly increasing, concave, and satisfies the single-crossing property: $u_{x\theta} \geq 0$ whenever $u_x \geq 0$. Further, $u(F(x,\theta))$ is supermodular in x for each θ and deterministic payoff $F : L \times \theta \to \mathbb{R}$, increasing in x. [1]

(A3) The dis-utility function $v : L \to \mathbb{R}$ is increasing in L.

(A4) Each C_t is strictly increasing in $|x|$, and minimized at zero: $C_t(0) = 0$.

(A5) The expected utility function $V(x, G) := \mathbb{E}_G[u(F(x,\theta))]$ is strictly quasi-concave, with a unique maximizer for each G.

Proposition 1 (Bounded Adjustment Path under Optimistic Beliefs)**.** *Suppose* (A1)–(A5) *hold, and let $G^2 \succ_{MLR} G^1$. Let $\underline{x} = \arg\max_x V(x, G^1)$ and $\bar{x} = \arg\max_x V(x, G^2)$ be the frictionless benchmarks. Then there exists a sequence $\{x_t\}_{t=1}^{\infty}$ solving the dynamic problem $\max_{\{x_t\}} \mathcal{J}(\{x_t\}; x_0, G^2)$ such that:*

$$\underline{x} \;\leq\; x_t \;\leq\; \bar{x}, \quad \textit{for all } t \geq 1.$$

[1] A sufficient primitive condition for this is: for all x, θ and $i \neq j$,

$$u''\big(F(x,\theta)\big) F_{x_i}(x,\theta) F_{x_j}(x,\theta) + u'\big(F(x,\theta)\big) F_{x_i x_j}(x,\theta) \;\geq\; 0.$$

Proof. The proof proceeds in two steps.

Step 1: Comparative statics of the frictionless benchmark. We first show that $\bar{x} \geq \underline{x}$ under the assumption that $G^2 \succ_{MLR} G^1$ and that the utility function u satisfies the single crossing property, i.e. $u_{x\theta} \geq 0$ whenever $u_x \geq 0$.
Fix the frictionless optimizer $\underline{x}$ under G^1, which satisfies:

$$V_x(\underline{x}, G^1) = \int_{\underline{\theta}}^{\bar{\theta}} u_x(\underline{x}, \theta)\, dG^1(\theta) = 0.$$

By continuity of u_x in θ, define:

$$\Theta := \{\theta \in [\underline{\theta}, \bar{\theta}] \ : \ u_x(\underline{x}, \theta) = 0\},$$

which is nonempty, closed, and bounded. Let $\underline{\theta}^* = \inf \Theta$. The assumption $u_{x\theta} \geq 0$ implies $x^*(\delta_\theta)$ is nondecreasing in θ. This further implies:

$$u_x(\underline{x}, \theta) < 0 \quad \text{for all } \theta < \underline{\theta}^*, \qquad u_x(\underline{x}, \theta) \geq 0 \quad \text{for all } \theta \geq \underline{\theta}^*.$$

Now suppose $G^2 \succ_{MLR} G^1$. Then, by Lemma 6, there exist points $\theta_1 \leq \theta_2$ and a nonnegative, increasing function $h : [\theta_1, \theta_2] \to \mathbb{R}$ such that:

$$G^2(\theta) = \int_{\theta_1}^{\theta} h(\xi)\, dG^1(\xi) \quad \text{for } \theta \in [\theta_1, \theta_2],$$

and G^2 assigns zero mass below θ_1, while G^1 assigns zero mass above θ_2.

Then:

$$\begin{aligned} V_x(\underline{x}, G^2) &= \int_{\underline{\theta}}^{\theta_1} u_x(\underline{x}, \theta)\, dG^2(\theta) + \int_{\theta_1}^{\theta_2} u_x(\underline{x}, \theta)\, dG^2(\theta) + \int_{\theta_2}^{\bar{\theta}} u_x(\underline{x}, \theta)\, dG^2(\theta) \\ &= \int_{\theta_1}^{\theta_2} u_x(\underline{x}, \theta)\, h(\theta)\, dG^1(\theta) + \int_{\theta_2}^{\bar{\theta}} u_x(\underline{x}, \theta)\, dG^2(\theta), \end{aligned}$$

since G^2 puts no mass below θ_1.

If $\underline{\theta}^* \leq \theta_2$, then $u_x(\underline{x}, \theta) \geq 0$ on $[\underline{\theta}^*, \bar{\theta}] \supseteq [\theta_2, \bar{\theta}]$, and $h(\theta) \geq 0$ by construction. Thus:

$$V_x(\underline{x}, G^2) = \int_{\theta_1}^{\theta_2} u_x(\underline{x}, \theta)\, h(\theta)\, dG^1(\theta) + \int_{\theta_2}^{\bar{\theta}} u_x(\underline{x}, \theta)\, dG^2(\theta) \ \geq \ 0.$$

By strict quasiconcavity of $V(x, G^2)$, it follows that:

$$\bar{x} \geq \underline{x}.$$

Step 2: Bounding the dynamic path (with costs). Let $\{x_t\}_{t=1}^{\infty}$ be one sequence that maximizes $\mathcal{J}(\cdot; x_0, G^2)$. It suffices to show that $\{\bar{x} \wedge (\underline{x} \vee x_t)\}_{t=1}^{\infty}$ also maximizes

$\mathcal{J}(\cdot; x_0, G^2)$, since $\underline{x} \le \bar{x} \wedge (\underline{x} \vee x_t) \le \bar{x}$ for each t.

Step 2(a): Showing $\{\hat{x}_t\}_{t=1}^{\infty} = \{\underline{x} \vee x_t\}_{t=1}^{\infty}$ is optimal. Given

$$\underline{x} \in \arg\max_{x \in L} \int_{\underline{\theta}}^{\bar{\theta}} u(F(x,\theta))\, dG^1(\theta),$$

we have

$$V(\underline{x}, G^1) \ge V(\underline{x} \wedge x_t, G^1).$$

By (A3), $u(F(\cdot,\theta))$ is supermodular in x for each θ, and integration preserves supermodularity[2]; hence

$$V(\underline{x}, G^1) + V(x_t, G^1) \le V(\underline{x} \vee x_t, G^1) + V(\underline{x} \wedge x_t, G^1)$$

which implies

$$V(\underline{x} \vee x_t, G^1) \ge V(x_t, G^1).$$

By the monotone comparative statics result of Athey (2002), under $G^2 \succ_{\text{MLR}} G^1$ and the single crossing property of $u(F(\cdot,\theta))$, this inequality is preserved:

$$V(\underline{x} \vee x_t, G^2) \ge V(x_t, G^2).$$

From this we obtain the explicit integral dominance

$$\int_{\underline{\theta}}^{\bar{\theta}} u(F(\underline{x} \vee x_t, \theta))\, dG^2(\theta) \ge \int_{\underline{\theta}}^{\bar{\theta}} u(F(x_t, \theta))\, dG^2(\theta) \quad \text{as well as} \quad u'(\cdot) > 0, \tag{21}$$

which gives $F(\underline{x} \vee x_t, \theta) \ge F(x_t, \theta)$ for all $t \ge 1$.

For adjustment costs, by Lemma 1 (Appendix G) in Dekel et al. (2025), for every $t \in \mathbb{N}$,

$$C_t(\underline{x} \vee x_t - \underline{x} \vee x_{t-1}) \le C_t(x_t - x_{t-1}).$$

Since C_t is monotone and $v'(\cdot) > 0$, we also have

$$\int_{\underline{\theta}}^{\bar{\theta}} v(C_t(\underline{x} \vee x_t - \underline{x} \vee x_{t-1}))\; dG^2(\theta) \;\le\; \int_{\underline{\theta}}^{\bar{\theta}} v(C_t(x_t - x_{t-1}))\; dG^2(\theta). \tag{22}$$

Combining (21) and (22) and using the concavity of u, we conclude that, for all $t \ge 1$,

$$\int_{\underline{\theta}}^{\bar{\theta}} \big[u\big(F(\underline{x}\vee x_t,\theta)\big) - v\big(C_t(\underline{x}\vee x_t - \underline{x}\vee x_{t-1})\big)\big]\, dG^2(\theta) \;\ge\; \int_{\underline{\theta}}^{\bar{\theta}} \big[u\big(F(x_t,\theta)\big) - v\big(C_t(x_t - x_{t-1})\big)\big]\, dG^2(\theta),$$

i.e.,

$$J(\underline{x} \vee x_t; \underline{x} \vee x_{t-1}, G^2) \ge J(x_t; x_{t-1}, G^2).$$

[2] If $h(x,\theta)$ is supermodular in x for every θ, then $H(x) = \int h(x,\theta)\, d\mu(\theta)$ is also supermodular. See, e.g., Topkis (1978), Thm. 2.6.1.

Hence $\{\underline{x} \vee x_t\}_{t=1}^{\infty}$ maximizes $\mathcal{J}(\cdot; x_0, G^2)$.

Step 2(b): Showing $\{\bar{x} \wedge \hat{x}_t\}_{t=1}^{\infty}$ is optimal. Let $\bar{x} \in \arg\max_x V(x, G^2)$. From $V(\bar{x}, G^2) \geq V(\bar{x} \vee \hat{x}_t, G^2)$ and supermodularity of V,

$$V(\bar{x} \wedge \hat{x}_t, G^2) \geq V(\hat{x}_t, G^2).$$

Provided u satisfies the single crossing property in $(x; \theta)$ and $G^2 \succ_{MLR} G^1$, Athey (2002) implies that V has the single crossing property in $(x; G)$; therefore

$$V(\bar{x} \wedge \hat{x}_t, G^2) \geq V(\hat{x}_t, G^2) \implies V(\bar{x} \wedge \hat{x}_t, G^1) \geq V(\hat{x}_t, G^1).$$

Equivalently, in integral form,

$$\int_{\underline{\theta}}^{\bar{\theta}} u(F(\bar{x} \wedge \hat{x}_t, \theta))\, dG^1(\theta) \;\geq\; \int_{\underline{\theta}}^{\bar{\theta}} u(F(\hat{x}_t, \theta))\, dG^1(\theta) \quad \text{and} \quad u'(\cdot) > 0, \tag{23}$$

which gives $F(\bar{x} \wedge \hat{x}_t, \theta) \geq F(\hat{x}_t, \theta)$ for all $t \geq 1$.
For the cost part, Lemma 1 (Appendix G) in Dekel et al. (2025) yields, for every $t \in \mathbb{N}$,

$$C_t(\bar{x} \wedge \hat{x}_t - \bar{x} \wedge \hat{x}_{t-1}) \;\leq\; C_t(\hat{x}_t - \hat{x}_{t-1}).$$

Since C_t is monotone and $v'(\cdot) > 0$, we obtain

$$\int_{\underline{\theta}}^{\bar{\theta}} v(C_t(\bar{x} \wedge \hat{x}_t - \bar{x} \wedge \hat{x}_{t-1}))\; dG^1(\theta) \;\leq\; \int_{\underline{\theta}}^{\bar{\theta}} v(C_t(\hat{x}_t - \hat{x}_{t-1}))\; dG^1(\theta). \tag{24}$$

Combining (23) and (24) gives, for all $t \geq 1$,

$$\begin{aligned} &\int_{\underline{\theta}}^{\bar{\theta}} \big[u(F(\bar{x} \wedge \hat{x}_t, \theta)) - v\big(C_t(\bar{x} \wedge \hat{x}_t - \bar{x} \wedge \hat{x}_{t-1})\big)\big]\, dG^1(\theta) \\ &\quad\geq \int_{\underline{\theta}}^{\bar{\theta}} \big[u(F(\hat{x}_t, \theta)) - v\big(C_t(\hat{x}_t - \hat{x}_{t-1})\big)\big]\, dG^1(\theta). \end{aligned}$$

Since $\{\hat{x}_t\}_{t=1}^{\infty}$ maximizes $\mathcal{J}(\cdot; x_0, G^2)$ and $\bar{x} \wedge \hat{x}_t$ agrees with $\hat{x}_t$ up to meet with $\bar{x}$, it follows that $\{\bar{x} \wedge \hat{x}_t\}_{t=1}^{\infty}$ also maximizes $\mathcal{J}(\cdot; x_0, G^2)$.
Therefore $\{\bar{x} \wedge (\underline{x} \vee x_t)\}_{t=1}^{\infty}$ is optimal, and the boundedness claim follows. □

The boundedness result in Proposition 1 provides a first qualitative restriction on optimal adjustment paths. However, boundedness alone does not ensure a well-structured dynamic profile of actions. To sharpen this characterization, we now strengthen the assumptions to exploit the supermodularity and curvature properties of the payoff function, which will allow us to establish that the optimal path can be taken to be monotone over time.

Proposition 2 (Monotone Adjustment Path under Strengthened Assumptions)**.** *In addition to the assumptions in Proposition 1, suppose further that the adjustment cost function $C(\cdot)$ does not change over time and $F(\cdot, \theta)$ is bounded on compact sets. Then, the sequence $\{x_t\}_{t=1}^{\infty}$ can be chosen to be monotone increasing. That is:*

$$\underline{x} \leq x_t \leq x_{t+1} \leq \bar{x}, \quad \textit{for all } t \geq 1.$$

Proof. The proof is by the standard monotonization / induction argument (cf. the monotonization step in Dekel et al. (2025)). Let $\{x_t\}_{t=1}^{\infty}$ be any optimal sequence for the dynamic problem under G^2; by Proposition 1 we may assume

$$\underline{x} \leq x_t \leq \bar{x} \qquad \text{for all } t \geq 1.$$

If $\{x_t\}$ is already nondecreasing, we are done. Otherwise we will construct, for every finite horizon T, a modified sequence x^T whose first T entries are nondecreasing and that achieves at least as large an objective as $\{x_t\}_{t=1}^{\infty}$. Letting $T \to \infty$ and using bounded convergence (or monotone convergence and the uniform bound coming from bounded payoffs) yields an optimal nondecreasing path.

Step 1 — Defining Monotonized Truncation. Fix $T \in \mathbb{N}$. Define the operator M_T on sequences in L such that $M_T x$ is a sequence with tth entry x_t for $t < T$ and $x_{t-1} \vee x_t$ for $t \geq T$. Define

$$X_t = x_1 \vee x_2 \vee \cdots \vee x_t \quad \text{for } t \in \mathbb{N}, \quad X_0 = \underline{x}.$$

Then, for the optimal sequence $\{x_t\}_{t=1}^{\infty} = x$, define

$$x^T = M_T M_{T-1} \cdots M_3 M_2 x \quad \text{for } T \geq 2.$$

The first T entries of this transformed sequence x^T are $X_1, X_2, X_3, \ldots, X_T$ and are monotonized since

$$\underline{x} \leq X_t \leq X_{t+1} \leq \bar{x} \quad \text{for any } t \in \mathbb{N}, \quad X_0 = \underline{x}.$$

Step 2 — To show x^T is optimal for each $T \in \mathbb{N}$. Suppose this argument holds and let $\mathcal{J}(\{x^T\}_{t=1}^{\infty}; X_0, G^2)$ be the optimal value for the dynamic problem. Since both sequences x^{∞} and x^T have the same first T entries, we write:

$$\mathcal{J}(\{x^T\}_{t=1}^{\infty}; X_0, G^2) \geq \mathcal{J}(\{x^{\infty}\}_{t=1}^{\infty}; X_0, G^2)$$

$$0 \geq \mathcal{J}(\{x^{\infty}\}_{t=1}^{\infty}; X_0, G^2) - \mathcal{J}(\{x^T\}_{t=1}^{\infty}; X_0, G^2)$$

$$0 \geq \sum_{t=T+1}^{\infty} \delta^t \int \Big\{ u(F(X_t, \theta)) - v(C(X_t - X_{t-1})) - \big[u(F(x_t^T, \theta)) - v(C(x_t^T - x_{t-1}^T)) \big] \Big\} dG^2(\theta).$$

From the monotonicity of C, we infer that

$$C(X_t - X_{t-1}) \leq C(x_t^T - x_{t-1}^T) \quad \text{for every } t \geq T+2,$$

$$C(X_{T+1} - X_T) \leq C(x_{T+1}^T - X_T).$$

Since $v'(\cdot) > 0$, we conclude

$$0 \geq \sum_{t=T+1}^{\infty} \delta^t \int \big(u(F(X_t, \theta)) - u(F(x_t^T, \theta))\big) dG^2(\theta). \tag{25}$$

For $t \geq T+2$ and each dimension i, if $X_{t,i} = X_{t-1,i}$, then $(X_t - X_{t-1})_i = 0$; if $X_{t,i} > X_{t-1,i}$, then $x_{t,i}^T = X_{t,i} \geq X_{t-1,i} \geq x_{t-1,i}^T$, so

$$0 \leq (X_t - X_{t-1})_i \leq (x_t^T - x_{t-1}^T)_i.$$

For $t = T+1$, if $X_{T+1,i} = X_{T,i}$, then $0 = (X_{T+1} - X_T)_i \geq (x_{T+1}^T - X_T)_i$; if $X_{T+1,i} > X_{T,i}$, then $X_{T+1,i} = x_{T+1,i}^T$.

Since $F(\cdot, \theta)$ is bounded on compact sets, expression (25) is uniformly bounded over T.[3] Taking $T \to \infty$ yields

$$0 \geq \mathcal{J}(\{x^\infty\}_{t=1}^\infty; X_0, G^2) - \mathcal{J}(\{x^T\}_{t=1}^\infty; X_0, G^2) \geq 0.$$

Hence x^∞ is optimal.

Step 3 — Induction Step: Monotone Optimality. We now show x^T is optimal for each $T \in \mathbb{N}$ using induction. The base case $T = 1$ is immediate.

Let us now suppose that for any fixed $T \in \mathbb{N}$, $x^T = \{x_t\}_{t=1}^\infty$ is optimal. All that remains to show is that $x^{T+1} = M_{T+1} x^T$ is also optimal.

Let $\{\tilde{x}_t\}_{t=1}^\infty$ be a sequence with tth entry

$$\tilde{x}_t = \begin{cases} x_t & t < T, \\ x_t \wedge x_{t+1} & t \geq T. \end{cases}$$

Since x^T is optimal and $\{\tilde{x}_t\}$ share its first $T-1$ entries, we have:

$$\mathcal{J}(\{x_t\}_{t=1}^\infty; X_0, G^2) \geq \mathcal{J}(\{\tilde{x}_t\}_{t=1}^\infty; X_0, G^2) \tag{26}$$

[3] Since both X_t and x_t^T are bounded between the long run benchmarks $\underline{x}$ and $\bar{x}$ for every $t \in \mathbb{N}$, there is $K > 0$ such that $F(X_t, \theta) - F(x_t, \theta) \geq -2K$ for all t, so F is bounded in L below by $-2K/(1-\delta)$

This can be rewritten as a single inequality with aligned sums:

$$\begin{aligned}&\sum_{t=T}^{\infty}\delta^{t-T}\left[\int\big(u(F(x_t,\theta))-u(F(x_t\wedge x_{t+1},\theta))\big)dG^2(\theta)\right]\\&\quad-\sum_{t=T}^{\infty}\delta^{t-T}\left[\int\big(v(C(x_t-x_{t-1}))-v(C(x_t\wedge x_{t+1}-x_{t-1}\wedge x_t))\big)dG^2(\theta)\right]\geq 0\end{aligned}\tag{27}$$

(Note that $x_{T-1}\wedge x_T = X_{T-1} = x_{T-1}$, so $x_t = X_t$ for every $t \leq T$.)

From (A3), $u(F(\cdot,\theta))$ is supermodular in x. Then for every $t \geq T$:

$$\int\big(u(F(x_t\vee x_{t+1},\theta))-u(F(x_{t+1},\theta))\big)dG^2(\theta)\geq\int\big(u(F(x_t,\theta))-u(F(x_t\wedge x_{t+1},\theta))\big)dG^2(\theta)\tag{28}$$

Furthermore, from Appendix H, Dekel et al. (2025), for each $t \geq T$:

$$\begin{aligned}&\int\big(v(C(x_t\vee x_{t+1}-x_{t-1}\vee x_t))-v(C(x_{t+1}-x_t))\big)dG^2(\theta)\\&\leq\int\big(v(C(x_t-x_{t-1}))-v(C(x_t\wedge x_{t+1}-x_{t-1}\wedge x_t))\big)dG^2(\theta)\end{aligned}\tag{29}$$

Finally, combining (26), (28), and (29), and adjusting the summation indices, we obtain:

$$\begin{aligned}&\sum_{t=T+1}^{\infty}\delta^{t-(T+1)}\Big[\int(u(F(x_t,\theta))-u(F(x_t\wedge x_{t+1},\theta)))dG^2(\theta)\Big]\\&\quad-\sum_{t=T+1}^{\infty}\delta^{t-(T+1)}\Big[\int(v(C(x_t-x_{t-1}))-v(C(x_t\wedge x_{t+1}-x_{t-1}\wedge x_t)))dG^2(\theta)\Big]\geq 0\end{aligned}\tag{30}$$

By inspection, this shows that $\{x^{T+1}\}_{t=1}^{\infty} = M_{T+1}x^T$ satisfies:

$$\mathcal{J}(\{x^{T+1}\}_{t=T+1}^{\infty};X_T,G^2)\geq\mathcal{J}(\{x_t\}_{t=T+1}^{\infty};X_T,G^2)$$

Since x^T is optimal and x^{T+1} agrees with x^T in the first T entries, x^{T+1} is also optimal. By induction, the sequence can be chosen to be non-decreasing. □

5.3 Significance of Dynamic Results

This result generalizes the classical Le Châtelier principle to a dynamic stochastic environment. The agent gradually adjusts toward a long-run optimum, and more optimistic beliefs (in the MLR sense) lead to higher action paths over time. Crucially, this holds even in the presence of risk aversion and non-linear utility, thanks to the structural properties of expected utility under MLR shifts.

From an economic standpoint, this result helps explain gradual investment or policy adjustments under uncertainty: agents are pulled toward a more aggressive response

when their outlook improves, but adjustment costs make this process path-dependent and incremental. The dynamic Le Châtelier effect characterizes this directional pull over time.

5.4 Connection to Dekel et al. (2025)

Dekel et al. (2025) study dynamic optimization problems with time-invariant, deterministic payoffs and separable adjustment costs. Their central result shows that when the payoff function is supermodular and exhibits increasing differences, any optimal sequence of actions can be monotonized—that is, replaced by a non-decreasing path—without loss of value.

Our framework retains the lattice-theoretic structure they exploit but differs in several key respects. First, we work in an environment with *expected utility* payoffs, where the decision maker chooses actions after observing only a *distribution* of an uncertain parameter $\tilde{\theta}$, rather than its realized value. Under suitable conditions, expected utility preserves supermodularity and increasing differences, allowing us to adapt their monotonization argument to a stochastic setting. This connection makes monotone comparative statics (and in particular the role of monotone likelihood ratio (MLR) shifts in $\tilde{\theta}$) directly relevant: MLR shifts capture systematic changes in the distribution of states that naturally arise in information acquisition or Bayesian learning problems.

Second, the source of uncertainty differs substantially. In our setup, uncertainty enters directly through an *unknown state variable* that affects payoffs, bringing the analysis closer to the earlier literature on comparative statics under risk and uncertainty. In contrast, Dekel et al. (2025) introduce uncertainty through *adjustment costs*, with the underlying state fixed. This distinction changes the interpretation of monotonicity results: in our case, they describe how optimal actions respond to stochastic states; in theirs, how actions adjust over time in response to cost shocks.

Finally, our current results rely on assuming that $u(F(\cdot))$ is supermodular. While sufficient for applying the global Le Châtelier principle, this is a strong restriction—concave transformations of supermodular functions are supermodular only under stringent conditions. A promising direction for future work is to identify weaker conditions on preferences and technology(such as quasi-supermodularity and single crossing) that still yield the desired monotonicity and concavity properties, potentially bypassing the need for full supermodularity of $u(F(\cdot))$.

We now turn to the "value of information" itself, reframing our dynamic setting in the uncertainty-as-constraint framework so as to connect local comparative statics to global curvature properties.

6 On the Global Convexity of Value of Information

Motivation and Background

It is well understood that perfectly accurate information has non-negative value. More subtle is the question of how this value changes as the uncertainty faced by a decision-maker varies. A natural conjecture is that greater uncertainty should increase the expected value of information, but this relationship is not guaranteed by general maximization principles without further structure on preferences or technology.
Plummer (1985), following Silberberg's Generalized Envelope Theorem (1971) and the local Le Châtelier Principle (LCP) (1947), provided a clean proof of local convexity of the value of information in the uncertain parameter. His insight was to treat uncertainty as a constraint on the decision problem, identify a just-binding point where the constraint does not alter the optimal choice, and then invoke the LCP to show that, in a neighborhood of this point, the ex post value of information is convex. Combined with Rothschild-Stiglitz (1978), this shows that a mean-preserving spread in uncertainty locally increases the expected value of information.
However, the LCP is essentially a local property. Plummer's argument guarantees convexity only in the immediate vicinity of a binding point; away from it, reversals in comparative statics or discontinuities in payoffs can generate concave regions in the value function. Samuelson's early investigations into the 'global' LCP demonstrated that, in many economic environments, such global monotonicity does not need to hold. In this section, I seek conditions under which the value of information is globally convex in the underlying uncertainty, so that any mean-preserving spread will raise its expected value. To do so, I build on the formulation of Milgrom and Roberts (1994) of the global Le Châtelier principle, which identifies environments where the result of comparative static dominance of the LCP extends beyond the local neighborhood. By embedding Plummer's uncertainty as constraint framework within the Milgrom–Roberts global theory, I derive sufficient conditions, interpretable in behavioral terms, that ensure the value of information retains convexity over the entire domain of the uncertain parameter.

To formalize these ideas, we adopt the "uncertainty-as-constraint" framework used by Plummer (1985). This approach has two advantages for our purposes. First, it allows us to express the value of information in terms of two related optimization problems: one in which the uncertain parameter is known before the choice of actions, and one in which it is not. Second, it makes explicit the just-binding point at which the presence of uncertainty does not affect the optimal choice, enabling a clean application of the Le Châtelier Principle. Plummer's key result was that, under the usual concavity and differentiability assumptions, the ex post value of information is locally convex in the uncertain parameter in a neighborhood of such a point. We now set up these problems in a form suitable for extending his local convexity result toward conditions for global convexity.

6.1 Setup and Notation

We begin by formalizing the decision environment. The decision maker (DM) chooses a vector of actions

$$x = (x_1, x_2, \ldots, x_n) \in X \subseteq \mathbb{R}^n$$

in order to maximize

$$V = F(x_1, x_2, \ldots, x_n; \theta_1, \theta_2, \ldots, \theta_n),$$

where the function $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ represents the DM's payoff. We assume F is twice continuously differentiable in x and jointly continuous in (x, θ). Moreover, for every parameter vector θ in the relevant domain, $F(\cdot; \theta)$ is strictly concave in x.

The feasible set X is assumed to be non-empty, convex, and compact. Under these standard conditions, each maximization problem below has a well-defined solution, and the strict concavity of F ensures that the solution is unique. The negative definiteness of the Hessian $D^2_{xx}F$ at the optimal point provides the second-order condition for a maximum.

Uncertainty enters the problem through the parameter $\tilde{\theta}_1$, which is a real-valued random variable with cumulative distribution function $G(\cdot)$ and support $\Theta_1 \subset \mathbb{R}$. The remaining parameters $\theta_2, \ldots, \theta_n$ are known and fixed. At some point in time, $\tilde{\theta}_1$ is realized, taking the value θ_1. Because of the strict concavity and smoothness of F, observing θ_1 prior to choosing x can only weakly improve the DM's payoff.

Case 1: θ_1 known before choice. If θ_1 is observed before the decision is made, the DM solves

$$\max_{x \in X} \; F(x; \theta_1, \theta_2, \ldots, \theta_n),$$

subject to the feasibility constraints in X. The strict concavity in x guarantees a unique optimal choice

$$x^*(\theta_1, \theta_2, \ldots, \theta_n),$$

satisfying the first-order conditions

$$\nabla_x F(x; \theta)\big|_{x = x^*(\theta)} = 0,$$

together with the negative definiteness of the Hessian $[F_{ij}]$. The associated indirect payoff is

$$V^*(\theta) \equiv F\big(x^*(\theta); \theta\big).$$

Case 2: θ_1 not known before choice. If the realization of θ_1 is not observed when x is chosen, the DM maximizes the expected payoff with respect to the distribution G:

$$\max_{x \in X} \; \mathbb{E}_{\tilde{\theta}_1} \left[F(x; \tilde{\theta}_1, \theta_2, \ldots, \theta_n) \right].$$

The optimal choice in this case,

$$x^E(\theta_2, \ldots, \theta_n, G),$$

solves the first-order condition

$$\nabla_x \mathbb{E}_{\tilde{\theta}_1} \left[F(x; \tilde{\theta}_1, \theta_2, \ldots, \theta_n) \right] \big|_{x=x^E} = 0,$$

and the expected Hessian is negative definite at x^E. The indirect payoff function is

$$V^E(\theta_2, \ldots, \theta_n, G) \equiv \mathbb{E}_{\tilde{\theta}_1} \left[F(x^E; \tilde{\theta}_1, \theta_2, \ldots, \theta_n) \right].$$

Ex post and ex ante value of information. For a given realization θ_1, the ex post value of information is defined as

$$\Phi^{\text{post}}(\theta_1; \theta_2, \ldots, \theta_n) = V^*(\theta_1, \theta_2, \ldots, \theta_n) - V^E\big(x^E; \theta_1, \theta_2, \ldots, \theta_n\big).$$

Because $x^*(\cdot)$ and x^E are uniquely determined, Φ^{post} is well-defined for each $\theta_1 \in \Theta_1$. The ex ante value of information is the expectation of the ex post value over the distribution of $\tilde{\theta}_1$:

$$\Phi^{\text{ante}}(\theta_2, \ldots, \theta_n, G) = \mathbb{E}_{\tilde{\theta}_1} \left[\Phi^{\text{post}}(\tilde{\theta}_1; \theta_2, \ldots, \theta_n) \right].$$

Plummer's contribution was to reinterpret the "unknown-θ_1" problem as the "known-θ_1" problem with an additional constraint: namely, that x must be chosen independently of θ_1. At a *just-binding point*—where this restriction does not alter the optimal choice—Silberberg's Generalized Envelope Theorem and the Le Châtelier Principle imply that V^{post} is locally convex in θ_1. In the next section, we build on this structure to examine conditions under which convexity can hold globally.

6.2 Reformulation as a Constrained Problem

Following Plummer's approach, the case in which θ_1 is unobserved can be represented as the case with θ_1 observed *plus* an additional constraint. Specifically, suppose the optimal choice under uncertainty is $x^E(\theta_2, \ldots, \theta_n, G)$ from the previous section. When θ_1 is in fact observed before choice, impose the restriction that the first $r \leq n$ components of x must be held fixed at the corresponding components of x^E, independent of θ_1. The remaining $(n - r)$ components may adjust freely.

Formally, consider the constrained problem:

$$\begin{aligned} \max_{x \in X} \quad & F(x_1, \ldots, x_n; \theta_1, \ldots, \theta_n) \\ \text{s.t.} \quad & x_i = x_i^E(\theta_2, \ldots, \theta_n, G), \quad i = 1, \ldots, r. \end{aligned} \tag{31}$$

Let $x^r(\theta_1, \ldots, \theta_n, G)$ denote the solution to (31), and define the associated indirect

payoff function

$$V^r(\theta_1, \ldots, \theta_n, G) \equiv F\big(x^r(\theta_1, \ldots, \theta_n, G); \theta_1, \ldots, \theta_n\big).$$

The superscript "r" indicates that r components of x are restricted to be independent of θ_1.

Value of information in constrained form. In this formulation, the *ex post* value of information can be written as

$$\Psi(\theta_1; \theta_2, \ldots, \theta_n, G) = V^*(\theta_1, \ldots, \theta_n) - V^r(\theta_1, \ldots, \theta_n, G).$$

At a just-binding point θ_1^{JB}, the constraint in (31) is slack and hence

$$V^*(\theta_1^{\mathrm{JB}}, \ldots) = V^r(\theta_1^{\mathrm{JB}}, \ldots), \quad \Psi(\theta_1^{\mathrm{JB}}) = 0.$$

Plummer's local convexity result then follows from applying Silberberg's Generalized Envelope Theorem to V^* and V^r at θ_1^{JB} and invoking the Le Châtelier Principle.

Objective for global convexity analysis. Our goal is to obtain conditions under which $\Psi(\theta_1; \cdot)$ is convex *globally* on Θ_1, so that any mean-preserving spread in $\tilde{\theta}_1$ increases the expected value of information. In the remainder of this section, we work with the constrained formulation (31) to characterize such conditions.

Analysis Throughout, $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is C^2; for each parameter vector θ the fully flexible and constrained problems admit unique solutions $x^*(\theta)$ and $x^r(\theta, G)$, respectively; $F_{xx}(x, \theta)$ is symmetric and negative definite at these solutions, and the free-variable principal block $F_{xx}^{TT}(x, \theta)$ is negative definite. Write

$$\Psi(\theta_1) \;=\; V^*(\theta_1, \ldots, \theta_n) \;-\; V^r(\theta_1, \ldots, \theta_n, G),$$

and define, for any (x, θ),

$$Q(x, \theta) \;:=\; F_{x\theta_1}(x, \theta)^\top \left[F_{xx}(x, \theta)\right]^{-1} F_{x\theta_1}(x, \theta), \qquad Q_r(x, \theta) \;:=\; F_{x\theta_1}(x, \theta)^\top \left[F_{xx}^{TT}(x, \theta)\right]^{-1} F_{x\theta_1}(x, \theta).$$

By envelope and implicit differentiation,

$$\Psi''(\theta_1) \;=\; \Big(F_{\theta_1\theta_1}(x^*, \theta) - F_{\theta_1\theta_1}(x^r, \theta)\Big) \;-\; \Big(Q(x^*, \theta) - Q_r(x^r, \theta)\Big), \tag{32}$$

where, to lighten notation, (x^*, x^r) abbreviate $(x^*(\theta), x^r(\theta, G))$.

Lemma 7 (Block-Inverse (Schur) Dominance)**.** *Consider a symmetric matrix* $F_{xx}(x, \theta)$

partitioned according to fixed coordinates S and free coordinates T as

$$F_{xx}(x,\theta) = \begin{pmatrix} A & B \\ B^\top & C \end{pmatrix}.$$

Define the full inverse and the "reduced" inverse that ignores the fixed coordinates by

$$M(x,\theta) := \big[F_{xx}(x,\theta)\big]^{-1}, \qquad M_r(x,\theta) := \begin{pmatrix} 0 & 0 \\ 0 & \big[F_{xx}^{TT}(x,\theta)\big]^{-1} \end{pmatrix}.$$

If $F_{xx}(x,\theta) \prec 0$ and $C = F_{xx}^{TT}(x,\theta) \prec 0$, then the difference between the full and reduced inverses satisfies

$$M(x,\theta) - M_r(x,\theta) \preceq 0.$$

Proof. *Using the Schur complement $S := A - BC^{-1}B^\top$, the block-inverse formula gives*

$$M - M_r = \begin{pmatrix} S^{-1} & -S^{-1}BC^{-1} \\ -C^{-1}B^\top S^{-1} & C^{-1}B^\top S^{-1}BC^{-1} \end{pmatrix}.$$

For any vector $z = (u,v)$ partitioned conformably, we have

$$z^\top (M - M_r) z = (u - BC^{-1}v)^\top S^{-1}(u - BC^{-1}v) \le 0,$$

because $S \prec 0 \implies S^{-1} \prec 0$. This shows $M - M_r$ is negative semidefinite. □

Intuition. The lemma formalizes the idea that when inverting a negative definite matrix, "fixing" some coordinates (here, S) reduces the curvature seen along the remaining free coordinates (T). In other words, removing degrees of freedom can only decrease, or at most leave unchanged, the curvature in the directions that remain free. For more information concering Schur complements, see (Horn and Johnson, 2012, Section 0.7.3).

Proposition 3 (Matrix sufficient condition for global convexity)**.** *If, for all realizations θ_1 in the parameter region,*

$$Q\big(x^*(\theta),\theta\big) \le Q_r\big(x^r(\theta),\theta\big) \quad \text{and} \quad F_{\theta_1\theta_1}\big(x^*(\theta),\theta\big) \ge F_{\theta_1\theta_1}\big(x^r(\theta),\theta\big),$$

then $\Psi''(\theta_1) \ge 0$ for all θ_1, i.e. Ψ is globally convex in θ_1. Proof. *Substitute the two inequalities into* (32)*.* □

While Proposition 3 gives a direct curvature-based route to global convexity, it can sometimes be difficult to verify the matrix inequality in practice. The next result provides a more tractable sufficient condition based on monotone comparative statics (MCS) arguments.

Proposition 4 (MCS route to global convexity)**.** *If F is supermodular in (x,θ_1), $F_{x\theta_1}(x,\theta)$ is componentwise nondecreasing in x, and the "uncertainty-as-constraint" feasible set is a*

sublattice in the free coordinates, then

$$Q\big(x^*(\theta),\theta\big) \;\le\; Q_r\big(x^r(\theta),\theta\big)$$

for all θ_1. By Proposition 3, Ψ is globally convex in θ_1.

Proof. By supermodularity of F in (x,θ_1) and the lattice structure of X, the Global Le Châtelier Principle (Milgrom–Roberts) implies that relaxing the constraints (i.e. moving from the constrained to the fully flexible problem) weakly increases the optimizer in the lattice order: $x^*(\theta) \succeq x^r(\theta)$ for every θ. Because $F_{x\theta_1}(x,\theta)$ is componentwise nondecreasing in x, we have the entrywise inequality

$$F_{x\theta_1}\big(x^*(\theta),\theta\big) \;\succeq\; F_{x\theta_1}\big(x^r(\theta),\theta\big).$$

Let $H(x,\theta) := -F_{xx}(x,\theta)$ and $H^{TT}(x,\theta) := -F^{TT}_{xx}(x,\theta)$; by the standing concavity assumptions these matrices are positive definite. Write

$$\widetilde{Q}(x,\theta) := F_{x\theta_1}(x,\theta)^\top\, H(x,\theta)^{-1}\, F_{x\theta_1}(x,\theta), \qquad \widetilde{Q}_r(x,\theta) := F_{x\theta_1}(x,\theta)^\top \begin{pmatrix} 0 & 0 \\ 0 & H^{TT}(x,\theta)^{-1} \end{pmatrix} F_{x\theta_1}(x,\theta),$$

so that the quadratic terms appearing in (32) satisfy

$$Q(x,\theta) = -\,\widetilde{Q}(x,\theta), \qquad Q_r(x,\theta) = -\,\widetilde{Q}_r(x,\theta).$$

Lemma 7 (Schur-dominance) applied to $H(x,\theta)$ implies

$$H(x,\theta)^{-1} - \begin{pmatrix} 0 & 0 \\ 0 & H^{TT}(x,\theta)^{-1} \end{pmatrix} \;\succeq\; 0,$$

hence for any fixed x and any vector v,

$$v^\top H(x,\theta)^{-1} v \;\ge\; v^\top \begin{pmatrix} 0 & 0 \\ 0 & H^{TT}(x,\theta)^{-1} \end{pmatrix} v,$$

i.e. $\widetilde{Q}(x,\theta) \ge \widetilde{Q}_r(x,\theta)$. Equivalently,

$$Q(x,\theta) \le Q_r(x,\theta) \qquad \text{for every } x.$$

Now use the optimizer ordering $x^*(\theta) \succeq x^r(\theta)$ together with the componentwise monotonicity of $F_{x\theta_1}$ and the assumed monotonicity (in Loewner order) of $H^{TT}(x,\theta)^{-1}$. These imply

$$\widetilde{Q}_r\big(x^*(\theta),\theta\big) \;\ge\; \widetilde{Q}_r\big(x^r(\theta),\theta\big),$$

so combining the previous inequalities yields

$$\widetilde{Q}\big(x^*(\theta),\theta\big) \;\geq\; \widetilde{Q}_r\big(x^*(\theta),\theta\big) \;\geq\; \widetilde{Q}_r\big(x^r(\theta),\theta\big).$$

Changing sign back to the Q notation gives

$$Q\big(x^*(\theta),\theta\big) \;\leq\; Q_r\big(x^r(\theta),\theta\big),$$

Finally, if F is also convex in θ_1, then

$$F_{\theta_1\theta_1}(x^*(\theta),\theta) \geq F_{\theta_1\theta_1}(x_r(\theta),\theta)$$

so that both conditions of Proposition 3 are satisfied. Therefore Proposition 3 applies and Ψ is globally convex in θ_1. □

Comment. Proposition 4 shows that supermodularity of F in (x,θ_1), combined with lattice structure and mild curvature regularity, is sufficient to deliver the matrix inequality needed in Proposition 3. In environments with x-invariant $F_{\theta_1\theta_1}$ (e.g. linear-in-parameter specifications), the direct-curvature term in (32) drops out and supermodularity alone, via GLCP, ensures global convexity of the value of information.

6.3 Concluding Remarks on Global Convexity

In sum, global convexity of Ψ provides a powerful lens for understanding the value of flexibility and information. While local convexity near the just-binding point is guaranteed by classical envelope results, establishing global convexity shows that the benefits of adaptability grow non-linearly across the entire state space. Under economically meaningful conditions, such as monotone optimal responses driven by supermodularity or increasing differences, we can ensure that $\Psi(\theta_1)$ rises as one moves away from the binding point, yielding robust comparative statics and ruling out diminishing returns to flexibility.

Looking forward, an important direction is to identify substantially weaker conditions on the objective function or the underlying problem structure under which global convexity still holds. Doing so could broaden the applicability of these insights, linking them more directly to real-world decision-making under uncertainty, information design, and option value problems.

The results on global convexity not only clarify when flexibility and information are most valuable, but also point toward broader questions. In particular, they motivate the search for weaker, more general conditions under which similar comparative statics and directional predictions hold. These insights set the stage for a final discussion of the overall implications, potential extensions, and open directions for future work.

7 Conclusion and Potential Extensions

This report revisited the Le Châtelier principle through its classical, local form and modern global generalizations, extending the analysis to incorporate uncertainty, risk preferences, and dynamic adjustment costs. We moved from Samuelson's bordered Hessians to lattice-theoretic monotone comparative statics, and from deterministic environments to stochastic and dynamic ones. The unifying thread is that relaxing constraints — whether technological, informational, or temporal — amplifies optimal responses, provided the problem's feedback structure is aligned.

Our results reaffirm the robustness of the principle. In smooth, concave settings, short-run attenuation follows directly from second-order conditions; in non-smooth or discrete settings, it emerges from supermodularity, quasi-supermodularity, and the single crossing property. Under uncertainty, both risk-neutral and risk-averse agents often display amplified long-run responses, with the form of risk preferences shaping the outcome. In dynamic models with adjustment costs, optimal adjustment paths remain monotone, and the eventual steady state under looser constraints dominates its short-run counterpart. However, these conclusions are based on strong assumptions: uniform complementarity/substitutability or their ordinal analogs, separable adjustment costs, and stationary environments. Weakening them without losing amplification remains an open challenge.

There are several directions for further research. First, one could seek weaker primitives for global amplification — for example, identifying minimal "region-by-region" complementarity conditions or non-separable adjustment costs that still guarantee a generalized Le Châtelier effect. Second, richer stochastic processes, including learning and belief updating, could be integrated into the dynamic framework to study how expectations themselves adjust over time. Third, and perhaps most intriguingly, the principle could be applied in strategic environments. Alexandrov and Bedre-Defolie (2017) show that pass-through and attenuation effects in games and market systems can be characterized in ways closely related to the Le Châtelier intuition. Extending the global and dynamic versions of the principle to supermodular games or other classes of strategic interaction could yield new insights into how constraints and flexibility shape equilibrium behavior when agents' actions are interdependent.

The Le Châtelier principle began as a qualitative observation in chemistry, migrated into economics as a local comparative statics theorem, and has since been reborn in global, ordinal, stochastic, and dynamic forms. Its enduring appeal lies in its simplicity of message paired with its breadth of application: systems resist change when constrained, and amplify change when freed. The challenge — and the opportunity — is to keep pushing the boundaries of where, and under what minimal conditions, that message continues to hold.

References

Aitken, A. (1933). On the evaluation of determinants, the formation of their adjugates, and the practical solution of simultaneous linear equations. *Proceedings of the Edinburgh Mathematical Society*, 3(3):207–219.

Alexandrov, A. and Bedre-Defolie, Ö. (2017). Lechatelier–samuelson principle in games and pass-through of shocks. *Journal of Economic Theory*, 168:44–54.

Athey, S. (2002). Monotone comparative statics under uncertainty. *The Quarterly Journal of Economics*, 117(1):187–223.

Athey, S. and Schmutzler, A. (1995). Product and process flexibility in an innovative environment. *The RAND Journal of Economics*, pages 557–574.

Baccelli, F. and Makowski, A. M. (1989). Multidimensional stochastic ordering and associated random variables. *Operations Research*, 37(3):478–487.

Batra, R. N. and Ullah, A. (1974). Competitive firm and the theory of input demand under price uncertainty. *Journal of Political Economy*, 82(3):537–548.

Dekel, E., Quah, J. K.-H., and Sinander, L. (2025). Comparative statics with adjustment costs and the le chatelier principle. *Econometrica*, 93(2):661–694.

Diamond, P. A. and Stiglitz, J. E. (1973). Increases in risk and in risk aversion.

Dietzenbacher, E. (1992). The measurement of interindustry linkages: Key sectors in the netherlands. *Economic Modelling*, 9(4):419–437.

Eichhorn, W. and Oettli, W. (1972). A general formulation of the lechatelier-samuelson principle. *Econometrica: Journal of the Econometric Society*, pages 711–717.

Epstein, L. G. (1987). The global stability of efficient intertemporal allocations. *Econometrica: Journal of the Econometric Society*, pages 329–355.

Etessami, K., Papadimitriou, C., Rubinstein, A., and Yannakakis, M. (2019). Tarski's theorem, supermodular games, and the complexity of equilibria. *arXiv preprint arXiv:1909.03210*.

Fujimoto, T. (1980). Global strong le chatelier-samuelson principle. *Econometrica: Journal of the Econometric Society*, pages 1667–1674.

Hadar, J. and Russell, W. (1978). Applications in economic theory and analysis. *Stochastic dominance*.

Hadar, J. and Seo, T. K. (1990). The effects of shifts in a return distribution on optimal portfolios. *International Economic Review*, pages 721–736.

Hennessy, D. A. (1997). The short-and long-run comparative statics of uncertainty. *Economics Letters*, 55(3):347–353.

Hoel, M. and Vislie, J. (1983). Supply functions for outputs and demand functions for inputs in the short and long run under uncertainty. *The Scandinavian Journal of Economics*, pages 53–60.

Horn, R. A. and Johnson, C. R. (2012). *Matrix Analysis*. Cambridge University Press, 2 edition.

Jamison, J. (2006). The le chatelier principle in lattices. *Economics Bulletin*, 3(2):1–9.

Karlin, S. and Rubin, H. (1956). The theory of decision procedures for distributions with monotone likelihood ratio. *The Annals of Mathematical Statistics*, pages 272–299.

Kusumoto, S.-I. (1976). Extensions of the le chatelier-samuelson principle and their application to analytical economics–constraints and economic analysis. *Econometrica: Journal of the Econometric Society*, pages 509–535.

Kusumoto, S.-I. (1977). Global characterization of the weak le chatelier-samuelson principles and its applications to economic behavior, preferences, and utility–" embedding" theorems. *Econometrica: Journal of the Econometric Society*, pages 1925–1955.

Lady, G. M. and Quirk, J. P. (2004). The le chatelier principle in a general setting. *Journal of Economic Theory*, 117(2):226–243.

Landsberger, M. and Meilijson, I. (1990). Demand for risky financial assets: A portfolio analysis. *Journal of Economic Theory*, 50(1):204–213.

Milgrom, P. and Roberts, J. (1990). Rationalizability, learning, and equilibrium in games with strategic complementarities. *Econometrica: Journal of the Econometric Society*, pages 1255–1277.

Milgrom, P. and Roberts, J. (1996). The lechatelier principle. *The American Economic Review*, pages 173–179.

Milgrom, P. and Shannon, C. (1994). Monotone comparative statics. *Econometrica: Journal of the Econometric Society*, pages 157–180.

Morishima, M. (1952). Demand functions with several independent goods. *Review of Economic Studies*, 20(1):1–26.

Nocetti, D. C. (2016). Robust comparative statics of risk changes. *Management Science*, 62(5):1381–1392.

Ormiston, M. B. (1992). First and second degree transformations and comparative statics under uncertainty. *International Economic Review*, pages 33–44.

Ormiston, M. B. and Schlee, E. E. (1993). Comparative statics under uncertainty for a class of economic agents. *Journal of Economic Theory*, 61(2):412–422.

Plummer, M. L. (1985). Uncertainty and the value of information: An application of the le chatelier principle. Working Paper 126, Federal Trade Commission, Bureau of Economics.

Quah, J. K.-H. and Strulovici, B. (2009). Comparative statics, informativeness, and the interval dominance order. *Econometrica*, 77(6):1949–1992.

Rothschild, M. and Stiglitz, J. E. (1978). Increasing risk: I. a definition. In *Uncertainty in economics*, pages 99–121. Elsevier.

Samuelson, P. (1963). Structure of a minimum equilibrium system, in "essays in economics and ž. econometrics: A volume in honor of harold hotelling"rw pfouts, ed., university of ž.

Samuelson, P. A. (1947). *Foundations of Economic Analysis*. Harvard University Press, Cambridge, MA.

Samuelson, P. A. (1949). The le chatelier principle in linear programming. Research Memorandum RM-210, RAND Corporation, Santa Monica, CA. U.S. Air Force Project RAND report; 18 pp.

Samuelson, P. A. (1960). An extension of the lechatelier principle. *Econometrica: Journal of the Econometric Society*, pages 368–379.

Samuelson, P. A. (1972). Maximum principles in analytical economics. *The American Economic Review*, 62(3):249–262.

Sandmo, A. (1971). On the theory of the competitive firm under price uncertainty. *The American Economic Review*, 61(1):65–73.

Shannon, C. (1995). Weak and strong monotone comparative statics. *Economic Theory*, 5:209–227.

Silberberg, E. (1971). The le chatelier principle as a corollary to a generalized envelope theorem. *Journal of Economic Theory*, 3(2):146–155.

Silberberg, E. (2018). Le chatelier principle. In *The New Palgrave Dictionary of Economics*, pages 7707–7712. Springer.

Snow, A. (2000). Lechatelier effects for the competitive firm under price uncertainty. *Southern Economic Journal*, 66(3):715–728.

Suen, W., Silberberg, E., and Tseng, P. (2000). The lechatelier principle: the long and the short of it. *Economic Theory*, 16:471–476.

Tarski, A. (1955). A lattice-theoretical fixpoint theorem and its applications. *Pacific Journal of Mathematics*, 5(2):285–309.

Topkis, D. M. (1978). Minimizing a submodular function on a lattice. *Operations research*, 26(2):305–321.

Viennot, X. (1986). Chains and antichains in partially ordered sets. *Advances in Mathematics*, 62(3):284–309.